5. The customary sequence of freshman courses in mathematics is well designed to develop the technical skill of students who intend to continue their mathematical studies. But since a large majority of students in mathematics will have no need for this technique in their later work, it would seem to be advisable to offer them, in colleges of liberal arts, a purely cultural course in mathematics which would emphasize the ideas and methods of mathematics, rather than technical drill. This course should be reserved strictly for students who have no intention of continuing in mathematics, and, for this and other reasons, should probably be open only to students who have completed the freshman year. This is in marked contrast to the practice at the University of Montana,[1] where the attempt is made to combine a cultural course with a course required as a preliminary to any other work in mathematics. It is rather in harmony with the earlier suggestions of Professor Lennes.[2] An outline for the course was suggested.

A. L. UNDERHILL, *Secretary.*

SUCCESSIVE GENERALIZATIONS IN THE THEORY OF NUMBERS[3]

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1. Contrasted with the laboriously slow evolution[4] of the complex number system now current in analysis, the corresponding development in arithmetic proper has been extraordinarily rapid. Possibly all the scientific progress of the race from prehistoric times to the Nineteenth Century is reflected in the successive extensions and perfections of the usual number system of analysis, culminating in the complex variable and hypercomplex numbers, for each successful attempt to think clearly about nature has left its impress on mathematical analysis and therefore ultimately upon the numbers which analysis subjugates and fertilizes.

The dim beginnings of this long evolution are apparently lost beyond hope of recovery, unless it may be that psychology shall some day find a necessary basis for any self-consistent thinking, and we can only speculate on what induced the first mathematical philosopher to imagine that he knew what he was talking about when he glanced over his offspring and his bananas and

[1] N. J. Lennes, *A new type of freshman course in mathematics*, this Monthly, vol. 33 (1926), pp. 307–315. See also N. J. Lennes, *A Survey Course in Mathematics* (Harper and Bros., 1926).

[2] N. J. Lennes, *Mathematics for culture*, Educational Review, 1914.

[3] A lecture given by invitation of the Mathematical Association of America at the joint meeting with the American Mathematical Society, Columbus, Ohio, September 8, 1926.

[4] Readers of the Monthly in those States of the Union where "evolution" is anathema may substitute for this word "abracadabra," provided they define the latter sensibly.

ejaculated "36". It is conceivable that whatever this nameless but immortal primate thought he was concerned with would bear but little resemblance to the esoteric formulation of 1, 2, 3, · · · achieved by Frege and Russell with their "class of all classes equivalent to a given class"—an abstraction worthy of a Hindu mystic and yet a practical thing.

2. We have alluded to this ethereal definition, which not infrequently provokes the mathematically irreverent to ribald scorn, because it is precisely the concept of class inclusion which Dedekind, in a single flash of penetrating genius, perceived to be the essence of the theory of arithmetical divisibility. The connection, if any, between the Frege-Russell definition and the constant recurrence of the rudiments of the abstract algebra of classes and relations in all theories of algebraic numbers, and in fact in many other theories styled arithmetical by their originators, is not immediately apparent, and it might well repay closer attention.

3. The "numbers" with which much of modern arithmetic is concerned are far removed from the 1, 2, 3, · · · which custom deludes us into thinking we comprehend; yet, if Kronecker's vision of the potential arithmetization of all mathematical analysis be not an illusion, it must be possible to trace the class concept in the theory of numbers from its taproot in 1, 2, 3, · · · to its latest efflorescence in the theories of Prüfer, v. Neumann, and Speiser. This is a task which may be suggested to the mathematically minded philosopher and to the philosophically inclined mathematician, as a respite from their nightmares to circumscribe the ever-shifting quantum theory or sound the bottomless bog of the transfinite.

The unravelling of this tangled skein of pure logic and modernized numbers might well exercise the ablest logician, and it seems strange that mathematical fundamentalists (in the reputable sense of those who busy themselves with scientific foundations) should have neglected arithmetic, whose underlying structure is seemingly identical with that irreducible bed-rock which Boole called "laws of thought," and which today is renovated by, and associated with, the school of Whitehead and Russell, for the transient flux of theories "thrust into the world before their time but half made up."

The foundations of geometry have had their share of attention. Few creative philosophers of this generation would deny that the critical insight gained from a postulational examination of geometry has clarified their outlook on time no less than on space. Is it too much to hope that a like scrutiny of modern arithmetic will also yield its rich reward in a clearer perception of thought itself?

4. We have mentioned the amazing speed with which modern arithmetic reached maturity once the need for an extension of the classic theory of numbers became apparent; the lifetime of a single man might have spanned it all.

Before outlining briefly a few salient characteristics of this explosive growth, we may call attention to a distinction in the abstract natures of the respective generalizations of "number" sought and attained in the domains of analysis and arithmetic.

A true generalization Θ of a theory T would include T as an instance, either by appropriate specialization of the primary elements ("relata") of the theory or by a specific interpretation of the abstract relations between the elements, and T should not similarly include Θ, since otherwise T, Θ would be abstractly identical. The emphasis of the generalization may be either on the relata or on the relations, or it may be distributed over both. Instances of all these possibilities occur in the successive generalizations which finally yielded the field of complex numbers and later the several systems of hypercomplex numbers of modern algebra. Geometry also presents numerous examples.

Incidentally it may be pointed out that generalization and abstraction are radically different processes, with abstraction on the distinctly lower level. In abstraction we exhibit the structure of a theory devoid of everything but sufficient "marks" (both of relata and of relations) to show forth that structure. In generalization we construct a theory which implies the theory generalized but which is not implied by that theory.

There exists no true generalization of the theory of the natural numbers $1, 2, 3, \cdots$. Fragments only of what is known as the classic theory of numbers have been generalized in the above sense, which is usually, by common consent, taken to be substantially the only one worthy the title of generalization. Nor has abstraction, a useful first step toward generalization, been attempted in any systematic manner in arithmetic.

The distinction between the development of the numbers of analysis and those of arithmetic to which we referred a moment ago is this: in the arithmetical generalizations the chief emphasis seems to have been on the relations. Thus, in order to reach a workable theory of divisibility Dedekind was compelled to turn his back on intuition and to imagine a wider relation—that of class inclusion—which is indeed a true generalization of the concept of divisibility. In this he seems to have been followed by subsequent writers, certainly by Kronecker, some of whose theories at first sight bear but little resemblance to his.

There is another type of generalization which, however, is too hazy to be worth pursuing. It may be asked, why draw a distinction between the complex numbers of analysis and those of arithmetic? Are not the latter included in the former? The answer is of course affirmative, but it must be qualified by the obvious remark that arithmetic does not begin until integral elements are created—until atoms and molecules, as it were, each with a recognizable

individuality of its own, emerge from that chaotic smudge which analysis calls the continuum. Attempts to unite the continuous and the discrete in unholy wedlock have proved barren so far as arithmetic is concerned. If such efforts are not contrary to nature they certainly defy history. This of course does not refer to the far reaching applications of analysis to arithmetic—a totally different thing which does not concern us here.

5. In tracing the evolution of the theory of algebraic numbers it is customary to go back at least as far as Euler and Lagrange, both of whom frequently used several of the simpler processes of the modern theory, but neither of whom can be said to have shown any evidence that he suspected the existence of a new universe behind the few curious phenomena which caught his passing attention.

A clearer line of descent can be traced from Gauss. The usual historical reference to Gauss' connection with the theory of algebraic numbers points to his introduction of the complex integers $a+bi$ in his profound study of the law of biquadratic reciprocity.

It is perhaps significant, as has often been observed, that the arithmetical theory of complex numbers is not a necessity for the object which Gauss had in view, but merely a convenience. Can the like be maintained with regard to any existing application of the theory of algebraic numbers and ideals to the theory of the rational integers? And if so, precisely what is gained for rational arithmetic by the introduction of algebraic numbers or ideals beyond a certain compactness of statement?

Without here venturing an opinion on the last questions, we shall point out merely that the germ of the modern theory of ideals is implicit in the classic work of Gauss much earlier than in the memoirs on biquadratic residues. It appears in fact almost at the beginning of the *Disquisitiones Arithmeticae* in the familiar notion of congruence: two rational integers a,b are called congruent with respect to a third rational integer m when their difference is divisible by m, and this is written: $a \equiv b \bmod m$.

Let us examine for a moment this epoch-making definition which originated with Gauss. It yields much more than a suggestive notation pointing the way to innumerable fruitful analogies with the theory of algebraic equations. It does two things, each of them fundamental for the modern theory of numbers.

First, it segregates the infinity of rational integers into a finite number m of classes, two integers being put into the same or different classes according as they are congruent or incongruent modulo m.

This principle of reflecting the properties of an infinite totality of elements in a finite set of suitably chosen representatives permeates all arithmetical theories since Gauss. We have grown so accustomed to its use that we often take the principle for granted and perhaps fail to ascribe due credit to it as a

creative device. Once it has been applied in any given instance we see how inevitable has been the application. The impulse to make use of the obvious is however one of the rare things in mathematics.

An apposite illustration of this reluctance to use what lies close at hand is afforded by the very theories we are about to describe. A Dedekind ideal is an infinite linear set of a particular kind, yet it does not seem to have occurred to anyone until very recently that this infinite totality can be replaced with profit by a representative finite set. That such is apparently the case is evident from the "ideal systems" introduced by Speiser (1926) in his theory of ideals for the Dickson arithmetic of an algebra.

The second epochal conquest achieved by Gauss' definition of congruence is equally elementary, and again it is one of those obvious things which are seen for the first time only by the eye of genius. The statement $a \equiv 0 \bmod m$ can be read either "a is divisible by m" or "a is a member of the class of integer multiples of m," and it is the second of these that generalizes to yield the germs of the concepts of divisibility of ideals and of congruence with respect to an ideal modulus.

6. An outstanding difficulty, for which no comprehensive method of attack has been suggested, in the arithmetization of a given theory is the discovery of the correct twist to be given to the familiar relations of common arithmetic in order that the new relata shall yield relations abstractly identical with the particular relations being considered between rational integers. For example, how shall the concept of greatest common divisor be modified so that it may be significant for elements between which order relations (relations of greater or less) are without meaning?

We may suggest that a useful first step is the abstraction of the fundamental relations of common arithmetic, that is, their restatement in terms of "marks," and the comparison of the result with the like abstract formulations of such relations in the theory under investigation as are already well known and interesting. This implies as a preliminary the abstract formulation of the postulate systems of both theories.

In this project categorical systems of postulates or other technically refined sets are beside the point. What is wanted in each case is a workable set of postulates which will reveal at a glance those properties of the elements considered which are of mathematical importance and which it is desirable to abstract and, if possible, generalize. In short, the familiar must not be tortured out of all recognition into the bizarre in order to gain a doubtful logical perfection according to the prevailing whim of the moment among expert postulate makers, otherwise only an impossibly complicated machinery is acquired at the expense of mathematical power. If it be possible by selecting as basic some unusual property of the elements under consideration to reduce the

number of postulates defining them by one or two, the astute postulationalist will doubtless do so. The wily mathematician, on the other hand, with a different object in view, knows better than to blunder on his own scaffold and hang himself with his own rope. It is not here a question of right reasoning or of wrong, but of scientific expediency.

7. We have referred to the arithmetization of a theory, and although the sense in which the term is here used is plain by implication, it may be well to consider it briefly so that there shall be no misunderstanding.

In the first place "arithmetization" is used in what may appear to be a a broader meaning than that of Kronecker but which in the end may be formally equivalent to his.

Kronecker wished among other things to replace mathematical analysis by chains of finite operations upon the positive rational integers alone, and he showed specifically how the calculus of algebraic numbers can be included in such a program. This great project of Kronecker's, the subject of a lifelong dispute with Weierstrass who was of a diametrically opposite scientific temperament, is today of fresher interest than it has been since Kronecker's death, in view of the revolutionary work of Brouwer and his pupils. It begins to appear that Kronecker's prophecy to Weierstrass that, if he (Kronecker) himself should lack the time to push the matter through to a conclusion, his successors would, and see to it that analysis is founded rigorously on a strict arithmetical basis will be fulfilled.

Kronecker's program has never been cordially received, especially by the analysts nor, what is rather strange, by the professed prophets of mathematical rigor. The latter have objected to a thorough overhauling of their wares (including the Cantorian transfinite) on the ground that such a procedure would belong to metaphysics rather than to mathematics, and that rigor as pure logic demands it is a fantasy with which rigor in mathematics need have no traffic. Rigor, in other words, for some who dislike Kronecker's "mathematical anarchy," is on a par with virtue—a vaguely relative term whose intensity depends only upon the taste or distaste of the user. It is little less than sardonic justice that the present distress of the body analytic should have been induced by the execrable metaphysics which analysis swallowed in transfinite doses over Kronecker's protest.

Kronecker's arithmetization would make the positive rational integers central and supreme. The emphasis seems to be on the elements rather than on the relations between them, although the latter also are taken account of in the rejection of irrational and other interminable processes. If the emphasis be shifted from the relata to the relations, we may perceive another aspect of arithmetization, which is as follows.

Suppose it is possible from the given elements of a theory to construct new elements between which there subsist relations abstractly identical with certain relations in the theory of the rational integers. Then, to the extent of these relations, the theory of the new elements, whatever their nature, is abstractly identical with arithmetic. The problem in any case is to construct the new elements or to reconstruct the relations between the original so that an abstract arithmetical structure shall emerge.

The work of Dedekind abounds in striking illustrations of several of the points which we have mentioned. His early expositions are particularly valuable to present day workers because he has spared no pains to show us the subtle trains of reasoning which led him to his inspirations. Truly great, this mighty genius, unlike some others, was above clearing away the scaffolding before permitting the less gifted to view his perfected work. His preliminary bulletins on the theory of ideals are an unsurpassed laboratory in the method of mathematical discovery.

Returning for a moment by way of illustration to the G. C. D., let us observe how it is recast to make it capable of extension to domains in which order relations are meaningless. Evidently the following definition preserves all the properties of the G. C. D. with which we are familiar in common arithmetic, and yet it makes no explicit mention of "greatest" or of any other concept of magnitude: the G. C. D. of the positive rational integers a, b is that common divisor d of a, b which is such that, if δ is any common divisor of a, b then δ divides d. Similarly the L. C. M. of a, b is that common multiple m of a, b which is such that, if μ is any common multiple of a, b, then m is a multiple of μ.

Notice that these definitions, having isolated extensible properties of the G. C. D. and L. C. M., describe them wholly in terms of relations. Uniqueness is included in the definitions merely for brevity. In a given domain uniqueness is either proved to subsist or, if such be not the case, then each of the G. C. D. and L. C. M. is replaced by a class of divisors having the essential properties indicated.

Under these reconstructed definitions we may, for example, evolve an arithmetic either for classes or for relations, in which unique factorization, congruences, etc., of classes or relations are abstractly identical with the like for the positive rational integers. In the arithmetic of classes we may take the G. C. D. of two classes to be a definite one of their logical sum or their logical product, depending upon the definition of division; the L. C. M. is then the other. Division for classes may be defined in either of two ways: "a divides b," where a, b are classes, if (1) a includes b, or (2) a is included in b. The arithmetics developed on these respective definitions are of course distinct in their interpretations; abstractly they are identical. Two arithmetics exist for either classes or relations owing to the Pierce-Schroeder dualism in symbolic logic.

As another illustration, well known and easily accessible in elementary texts, we recall the theory of polynomials modulo a rational prime which is basic in Zolotareff's theory sketched later. In this theory there is unique factorization, etc., precisely as in the arithmetic of the rational integers or of ideals, although the interpretations are distinct. It is interesting in this theory of polynomials to observe the manner in which a large region of infinite rational arithmetic is imaged on a finite mirror.

We shall next state a few definitions and theorems from each of three widely different theories of ideals in order to suggest definite problems in the concluding sections. There is no attempt at completeness in what follows; we have endeavored to select similarly oriented phenomena from three distinct theories (with more space at our disposal we might easily have made it a dozen) which tend toward the same goal. Superficially these three (and there are many more) are violently unlike. It remains to be seen, from a complete abstract formulation, whether the differences go deeper than the surface. My own opinion, based on abstract formulations of several such theories, is that the apparent differences are principally variants of interpretations of one and the same body of pure (abstract) mathematics.

8. Dedekind's theory (of about 1876) belongs to the multiplicative half of arithmetic. Underlying the entire structure is the definition of integer in an enlarged sense. The theory of arithmetical divisibility occupies the central position. Arithmetical divisibility exists in a theory in the strict sense, namely in that of the rational integers, only when there is a law of unique factorization into indecomposable or prime integral elements. This in fact may be taken as the cardinal distinction between algebraic and arithmetical divisibility. It has even been proposed (by Cahen) to define arithmetic as that branch of algebra in which division is only exceptionally possible.

The goal in Dedekind's theory is the restoration of unique factorization by the introduction of properly defined elements (ideals) to replace the original elements (algebraic integers) for which unique factorization into primes fails. To be effective the new divisibility must be in at least one respect formally equivalent to the old—or nothing is gained by the invention of ideals, and the original problem of unique factorization is not solved but is replaced by another which has no significant relation to the original. We recall that two propositions are "formally equivalent" when each implies the other.

The following can be understood as an abstract proposition irrespective of the specific meanings later attached to the technical terms involved: if α, β are algebraic integers, α divides β when and only when the principal ideal (α) divides the principal ideal (β); moreover an ideal is uniquely the product of prime ideals (not necessarily nor in general principal). This is the link connecting factorization of algebraic integers with that of ideals. Evidently on

this basis a formal equivalence between the theories of divisibility of ideals and of algebraic integers, considered as principal ideals, can be established, for a given algebraic integer determines precisely one principal ideal. We shall see later that the desired end can be attained otherwise. First let us explain the terms used above.

9. A root x of an equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

in which the a's are rational numbers is called an algebraic number; if the a's are (rational) integers, x is an algebraic integer. It can be shown without difficulty that the sum, difference, product of two algebraic integers is an algebraic integer.

Irrespective of what interpretations shall be attached to the symbols *, † or to the phrase "* divides †," we shall write "* divides †" as follows: $*\,|\,\dagger$. Divisibility in the arithmetical sense of algebraic integers $\alpha, \beta, \gamma, \cdots$ is defined as for the rational integers: $\alpha\,|\,\beta$ when and only when an algebraic integer γ exists such that $\beta = \alpha\gamma$.

An algebraic integer α satisfies precisely one equation irreducible in the rational domain with leading coefficient unity (and, by preceding definitions, the remaining coefficients rational integers). The degree n of this irreducible equation satisfied by α is called the degree of α and of the field $K(\alpha)$ generated by α; the remaining roots of the equation are the conjugates of α. From $K(\alpha)$ is isolated the set $I(\alpha)$ of all algebraic integers in $K(\alpha)$, including those of degree 1, (the rational integers). The product of α and all its conjugates is the norm $N(\alpha)$ of α.

The problem is to study the laws of divisibility in $I(\alpha)$. In $I(1)$, the set of all rational integers, unique factorization into indecomposable integers (primes) is implied by the following theorem: if a prime divides the product of two integers it divides at least one of the factors. It is the breakdown of this fundamental theorem, in general, in $I(\alpha)$ which necessitates the introduction of ideals. A simple $I(\alpha)$ for which ideals are necessary is Dedekind's example $I(i\sqrt{5})$.

A unit is an element of $I(\alpha)$ which divides each element of $I(\alpha)$, the quotient being also in $I(\alpha)$; a decomposable element of $I(\alpha)$ is one which is the product of two elements of $I(\alpha)$ not both units; an indecomposable or prime element of $I(\alpha)$ is one which is not decomposable. Elements of $I(\alpha)$ differing only by unit factors are not distinguished in questions of divisibility. In general an element of $I(\alpha)$ is not uniquely the product of primes.

To restore unique factorization in $I(\alpha)$ Dedekind cast about for some simple construct of elements of $I(\alpha)$ for which the fundamental operations addition and multiplication are well defined. The class proved to be what was wanted, for classes of elements can be added and multiplied even when the individual

elements of the classes cannot. This choice is singularly appropriate in the case of $I(\alpha)$, for not only can sets (classes) of elements of $I(\alpha)$ be added and multiplied, but the like is true (in this instance) within the classes themselves. Even were the latter not the case, as happens in some arithmetical theories, we could still proceed, but the means for doing so would take us too far afield here. Notice that in the case of algebraic numbers multiplication and addition of elements, although abstractly identical with the like for classes of the same elements, are different operations from those for the classes in that their interpretations are distinct in the two instances. Such an abstract identity of relation but difference of interpretation characterizes many situations in general arithmetic; it is ultimately the logical distinction between intension and extension.

The class concept was introduced by Dedekind as follows. A set M of elements of $I(\alpha)$ which is closed under subtraction, and hence under addition and subtraction, is called a module. If M be such that, if β is any element of M, and γ is any element of $I(\alpha)$, then $\beta\gamma$ is an element of M, the module M is called an ideal. Hence an ideal of $I(\alpha)$ is closed under addition and subtraction and multiplication by elements of $I(\alpha)$.

As customary Dedekind ideals of a given $I(\alpha)$ are denoted by small German letters, $\mathfrak{a},\mathfrak{b},\mathfrak{c}, \cdots$.

We saw that the Gaussian concept of congruence is equivalent to a restatement of certain aspects of division in terms of class inclusion. Division in Dedekind's theory becomes class inclusion in the following way. If β, γ are any elements of $I(\alpha)$, and $\mathfrak{a}$ is any ideal of $I(\alpha)$, we say that

$$\beta \equiv \gamma \bmod \mathfrak{a}$$

only when the difference of β, γ is an element of $\mathfrak{a}$. This is otherwise expressed by the locution "$\mathfrak{a}$ divides, or contains, $\beta-\gamma$."

Before stating how unique factorization is restored, let us see how order relations are reintroduced into this theory. It is proved that the elements (algebraic integers) of $I(\alpha)$ fall into a finite number of classes with respect to any given ideal modulus $\mathfrak{a}$, two elements of $I(\alpha)$ being put into the same or different classes according as they are or are not congruent modulo $\mathfrak{a}$; the number $N(\mathfrak{a})$ of distinct classes is called the norm of $\mathfrak{a}$. Theorems in $I(1)$, such as Fermat's or Wilson's whose proofs in $I(1)$ depend ultimately upon order relations or upon enumeration, are in general extended to the ideals of $I(\alpha)$ by replacing the (rational integral) modulus m by a norm as above defined.

Returning now to divisibility, we shall state the method by which unique factorization of elements of $I(\alpha)$ is accomplished. It is shown first that each ideal $\mathfrak{a}$ of $I(\alpha)$ (or, what is the same for our purpose, of $K(\alpha)$) is defined by a finite number, say s, of elements $\alpha_1, \alpha_2, \cdots, \alpha_s$ of $I(\alpha)$, so that, γ being any element of $\mathfrak{a}$,

$$\gamma = \mu_1\alpha_1 + \mu_2\alpha_2 + \cdots + \mu_s\alpha_s,$$

where the μ's are in $I(\alpha)$. Consequently $\mathfrak{a}$ is represented by the symbol $(\alpha_1, \alpha_2, \cdots, \alpha_s)$, and $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_s\alpha_s$ is in $\mathfrak{a}$, where the λ's are any elements of $I(\alpha)$.

Hence, $\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_s)$, $\mathfrak{b} = (\beta_1, \beta_2, \cdots, \beta_t)$ being any two ideals of $I(\alpha)$ the ideal represented by the st products $\alpha_j\beta_k (j = 1, \cdots, s;\ k = 1, \cdots, t)$ is determinate. This ideal is called the product of $\mathfrak{a}$ and $\mathfrak{b}$ and is written $\mathfrak{ab}$. Such is the definition of multiplication of ideals.

Divisibility for ideals is defined as follows in terms of class inclusion: $\mathfrak{a}|\mathfrak{b}$ ('the ideal $\mathfrak{a}$ divides the ideal $\mathfrak{b}$') when and only when $\mathfrak{a}$ contains ($\equiv$ includes) $\mathfrak{b}$, that is, when and only when every element of $\mathfrak{b}$ is in $\mathfrak{a}$.

The definition is justified as more than a mere play on words by the theorem that if $\mathfrak{a} \mid \mathfrak{b}$, then there exists a unique ideal $\mathfrak{c}$, called the quotient of $\mathfrak{b}$ by $\mathfrak{a}$, such that $\mathfrak{b} = \mathfrak{ac}$, precisely as in rational arithmetic.

An ideal defined by the single element β of $I(\alpha)$ is called principal and is written (β); the principal ideal (1) divides every ideal $\mathfrak{a}$ of $I(\alpha)$, and it is the only ideal of $I(\alpha)$ having this property. Hence the unique unit ideal is (1).

Contrary (as usual) to what common sense might predict, an ideal divisor is "greater," meaning "more inclusive" than the dividend. That this is reasonable is seen thus. In $I(1)$ we write (m) for the principal ideal consisting of all rational integral multiples of the rational integer m. Then, for example, $2 \mid 6$ and $(2) \mid (6)$ are true propositions, and (2) is obviously more inclusive than (6).

An ideal $\mathfrak{p}$ is called prime if $\mathfrak{p}$ is not the product of two ideals both different from the unit ideal (1).

The fundamental theorem of Dedekind's theory states that an ideal is uniquely the product of prime ideals.

Since $\beta|\gamma$, where β, γ are any elements of $I(\alpha)$, is formally equivalent to $(\beta) \mid (\gamma)$, the theory has reached its goal and unique factorization is restored in $I(\alpha)$. It would be more accurate however to say that unique factorization in $I(\alpha)$ is "induced" by the like in the set of all ideals of $I(\alpha)$, since the "restoration" is achieved by means of an isomorphism between two relations, one of which concerns the original set of elements for which factorization was sought, while the other does not immediately refer to those elements but to classes of them. The abstract situation presented by this restoration is of very great interest.

When we come to Prüfer's theory we shall frequently require one detail of definition: if $\mathfrak{a}$ is any ideal of $I(\alpha)$, and β an element of $I(\alpha)$, then $\mathfrak{a} \mid \beta$ means $\mathfrak{a} \mid (\beta)$. Conversely, if β is an element of $\mathfrak{a}$, then $\mathfrak{a}$ is said to divide ($\equiv$ contain) β, and when convenient this is written $\mathfrak{a} \mid \beta$. Strictly, of course, $\mathfrak{a} \mid \beta$ is meaningless unless some conventional significance (as above) be assigned to it, since $\mathfrak{a}$ is an ideal and β is not, and we defined division by an ideal only when the dividend also is an ideal.

The part played in the foregoing by the algebra of classes or of relations is conspicuous, and it is the essence of Dedekind's generalization of the concept of divisibility. Two further instances of the importance of the class in this theory may be noticed, as they are simple and require no further definitions for their statement. The G. C. D. of any number of ideals is their logical sum; their L. C. M. is their logical product. The multiplicative properties of ideals based upon these definitions are abstractly identical with the like for the rational integers.

There are many other interesting formal equivalents between the multiplicative theory of numbers, the theory of classes or of relations, and the theory of Dedekind ideals, but these we must pass over with a bare reference to the important concept of a class of ideals and the number—the so-called class number—of ideal classes in a given field. The last suggests interesting developments in the algebra of classes, thus repaying a part of the debt of the theory of algebraic numbers to symbolic logic.

No satisfactory definition of the sum or difference of two Dedekind ideals was devised until very recently, and this new attempt has yet to be scrutinized. Hence, as stated before, a complete generalization of rational arithmetic is a thing of the future. This statement again, however, may require modification with respect to Prüfer's recent theory, but the explicit rounding out of the Dedekind theory has not yet been exhibited.

One point in the foregoing sketch may be noticed particularly. The concept of the product of two ideals may possibly be discarded or replaced by another in the final theory of arithmetic, as forecast by the work of Speiser on Dickson arithmetics, in which the Dedekind module appears to be fundamental. These questions however are still in the melting pot.

10. Dedekind's theory has several advantages over its great predecessor, Kummer's (of about 1857), of which we need here refer only to two. Kummer, the first to define ideal numbers and to restore unique factorization in a field where the fundamental theorem of arithmetic does not hold, considered only fields defined by roots of unity, and not a general algebraic number field as did Dedekind. Despite the fact that Kummer's introduction of ideals into arithmetic was beyond all dispute one of the greatest mathematical advances of the nineteenth century, his work now is commonly said to be chiefly of historical interest, being supplanted by the theories of Dedekind and others, although some of the leading workers on Fermat's last theorem still use Kummer's theory, at least as a point of departure for certain of their researches. The second advantage possessed by theories of the Dedekind type over those of the Kummer species, is that the former explicitly exhibit their ideals while the latter do not.

Abstractly, however, a theory of the Kummer type is of the highest interest. It would take us too far here to justify this remark in detail, but we shall endeavour to indicate where the epistemological interest of such a theory lies. Instances of the abstract situation presented by Kummer's theory are rare in mathematics, although they are the rule rather than the exception in theoretical physics—an observation to which we shall recur in a moment.

Kummer did not define an ideal as a thing in itself, but instead used relations expressed in terms of ideal numbers without concerning himself whether or not such numbers have an independent existence. Thus, if the algebraic integer α has always a certain property A, which in Kummer's theory is the property that α is a root of one or more congruences, α is said to be divisible by a determinate ideal corresponding to the property A.

The abstract situation here is identical with that exemplified in many theories of classical (and now largely discarded) theoretical physics, for instance the hydrodynamical analogies of Kelvin and Maxwell in electrodynamics. Physics never comes to grips with the reality which it believes it is investigating, but invariably describes that hypothetical reality by establishing a correspondence or isomorphism with an ideal construct whose imagined anatomy is supposedly familiar. Kummer's theory is one of the few complete mathematical analogues of this tenuous and somewhat mystical methodology.

Although it is now a historical relic, Kummer's theory has in one respect a singularly modern aspect which is less prominent in Dedekind's and in those of the latter's followers. The emphasis on the relation as the central concept of pure mathematics is a development of the present century. The method of Kummer's theory is purely relational, in contrast to Dedekind's which is largely concerned with the relata (ideals) existing only by implication in another form (ideal numbers) in Kummer's theory. In this respect Kummer's theory is potentially more general than Dedekind's because the ideal elements of Kummer's theory may be any whatever provided they satisfy the relations. There is thus in this respect a higher order of abstraction in Kummer's theory than in Dedekind's.

Before leaving this, we recall that Dedekind himself suggested an interpretation of a part of his theory more general than that now current. The elements of a Dedekind module as defined above belong to $I(\alpha)$, that is, they are algebraic integers. But obviously, as remarked by Dedekind, this restriction is fortuitous so far as modules quâ modules are concerned. The elements $\beta, \gamma, \delta, \cdots$ of a module may be any whatever provided only that they are such that for any pair of them there exists a unique "sum," where "addition" satisfies the commutative law and has a unique inverse and the "sum" is again in the module, so that a module is a set closed under an operation abstractly identical with algebraic addition.

To illustrate Dedekind's remark we may take as the elements of a module either classes or relations, and evidently a similar generalization of Dedekind ideals is feasible. The unique inverse of addition is given by the supplementary class.

11. Intermediate in type between Kummer's theory and Dedekind's is that of Zolotareff (of about 1880), which also has some features in common with the most recent theories of algebraic numbers. Zolotareff's theory, as will be seen from the following sketch of its basic concepts, is abstractly of a structure similar to Kummer's in that its ideal numbers are relational rather than explicit. Again the algebra of classes is fundamental for the arithmetic, but in a less obvious way than in Dedekind's theory.

The algebraic aspects of Zolotareff's theory derive from Galois, and the entire fabric, attractive from several points of view, has an advantage over some other theories of algebraic numbers in that every indicated operation can be performed in a finite number of steps without any tentative process whatever. This advantage it shares with another, better known theory, that of Kœnig, which carries to reasonable completion Kronecker's extension of Dedekind's theory. The gain in each case however is perhaps only illusory, as the actual execution of the details —a matter which writers on this branch of mathematics usually disdain—may in practice prove to be prohibitively prolix. In short the arithmetical millenium has not yet arrived either in the rational or in the algebraic theory.

12. Zolotareff starts with an algebraic number field $K(\alpha)$ of degree n, and precisely as in Dedekind's theory, isolates the totality $I(\alpha)$ of integers in $K(\alpha)$. Let $\Pi \equiv \Pi(x)$ be the polynomial in x, algebraically irreducible in $K(1)$, whose vanishing defines $K(\alpha)$, the coefficient of x^n in Π being 1 and the remaining coefficients rational integers. Then, in this theory, the rational prime p is defined to be prime in $K(\alpha)$ if Π is irreducible modulo p. All remaining rational primes are classified as composite in $K(\alpha)$.

It is proved readily that if p is prime in $K(\alpha)$ then p divides the product of two or more integers of $K(\alpha)$ when and only when p divides one of the factors. The theory thus satisfies one of the cardinal requisites of any strict arithmetical theory.

Consider next the rational prime p for which Π is reducible modulo p, and let $V_j (j=1, \cdots, h)$ be the factors of Π modulo p, of the respective degrees $m_j (j=1, \cdots, h)$, so that

$$V_1^{m_1} V_2^{m_2} \cdots V_h^{m_h}$$

is the resolution of Π into powers of distinct (polynomial) factors irreducible modulo p. Then this rational prime p is said to be composite in $K(\alpha)$ and to have the prime ideal factor V_j precisely m_j times $(j=1, \cdots h)$. The factors V_j are said to belong to p.

If now $f(x_0)$ is an element of $I(\alpha)$, we say that $f(x_0)$ is divisible by a factor of p belonging to V if $f(x)$ is divisible modulo p by V. Set

$$W = V_1^{m_1} \quad \cdots V_k^{m_k},$$
$$W_1 = V^m V_2^{m_2} \cdots V_k^{m_k},$$
$$\cdots\cdots\cdots\cdots$$
$$W_k = V^m V_1^{m_1} \cdots V_{k-1}^{m_{k-1}},$$

(where in W_i the factor V_i is omitted), and let λ be a positive rational integer. Then, attending to the typical exponent m, we set $\lambda = km + r (0 \leqq r < m)$, and say that $f(x_0)$ has an ideal factor, belonging to V, precisely λ times (or of multiplicity λ), provided

$$f(x)V^{m-r}W^{k+1} \equiv 0 [\text{modd } p^{k+1}, \Pi],$$

is true, and

$$f(x)V^{2m-r-1}W^{k+2} \equiv 0 [\text{modd } p^{k+2}, \Pi]$$

is false. Similarly for other ideal factors and their multiplicities.

On these definitions, theorems abstractly identical with the fundamental ones of rational arithmetic are proved with comparative ease—certainly after a shorter chain of subsidiary theorems than is required in the usual presentation of Dedekind's theory after Hurwitz. Among these we note only the following.

If $f(x_0)$ is an arbitrary complex integer (element of $I(\alpha)$) which contains the ideal factor of the rational prime p belonging to V exactly λ times, and if $\psi(x_0)$ does not contain this factor, then $f(x_0)\psi(x_0)$ contains this factor precisely λ times.

There is unique factorization of elements of $I(\alpha)$ into prime ideal factors (in Zolotareff's sense).

The G. C. D. and L. C. M. are defined in a manner abstractly identical with that of rational arithmetic. Thus two elements of $I(\alpha)$ are co-prime if they have no common factors; their G. C. D. is the product of their common ideal factors, etc.

Zolotareff's theory refers to any algebraic number field. Its scope therefore is coextensive with that of the Dedekind theory. The special case of Zolotareff's theory in which the field is generated by a root of unity is easily seen to yield the Kummer theory.

We have given enough of this theory to satisfy our immediate purpose here, namely to show that unique factorization can be restored in at least two ways, and that the algebra of symbolic logic underlies the entire structure. Dedekind, Kronecker and others have sketched a third way of restoring unique factorization, that of adjunction to the original field, but this is not very attractive and it has not been followed out completely.

The class concept in Zolotareff's theory again is fundamental, being concealed in the notion of reducibility modulo a rational prime and in the definition of ideal factors.

Before leaving Zolotareff's ideas, we mention an easy extension of his theory which does not seem to occur in the literature: the rational prime p may be replaced by a prime ideal $\mathfrak{p}$, as in the theory of reducibility, unique factorization, etc., modulo p passes over at once to an identical theory modulo $\mathfrak{p}$. The resulting theorems of course belong to a domain different from that of Zolotareff's investigations.

We pass on to another method which has been proposed recently (1925, 1926). This seems to be the most inclusive of all, as will be seen from the summary. The substratum of algebraic logic underlying this theory and the abstract identity of its fundamental constructs with those of the theory of classes or relations is as strikingly apparent as in Dedekind's ideals. It is most clearly seen in v. Neumann's presentation, which we have followed here.

13. Prüfer's theory (of 1925) and its modifications by v. Neumann (1926) aim to be the most comprehensive yet devised for algebraic numbers.

A set of elements closed under multiplication and division is said to form a ray; a set closed under addition, subtraction and multiplication constitutes a ring. Hitherto, as already noted, all attempts to define addition or subtraction for Dedekind ideals have proved unsatisfactory. It is one of the features of Prüfer's theory that the new "integral ideals" which it introduces form a ring. These ideal numbers are an extension of the given algebraic number field, completing it, in a manner similar to that in which Dedekind ideals complete the rays of the field by restoring the laws of rational arithmetic.

That part of the theory which concerns multiplicative properties yields, upon the proper specializations, the theories of Kummer, Dedekind, and Kronecker. Moreover there exists in Prüfer's theory a unique additive decomposition for all ideal numbers, reminiscent of the direct sum in linear algebras, a feature not found in any previous theory. Further, the new theory can be carried over to the theory of algebraic functions of a single variable, but not to the like for more than one variable, an analogy, apparently, with the Dedekind-Weber theory in the same direction. It would be of great interest to see whether this theory is capable of restoring unique factorization in a Dickson arithmetic—the aim of Speiser's entirely different theory.

A new result of a different order states that the theory of relative fields is rendered superfluous. This is indeed good news if it be more than an existence theorem. Briefly the argument whereby this conclusion is reached is as follows. In Dedekind's theory as sketched above and as always presented in its rudiments, we took as the point of departure the rational number field and defined algebraic numbers in relation to it by postulating that the coefficients of the equation defining the field were rational numbers. The immediately suggested generalization of investigating any given algebraic number field in relation to any other, not necessarily the rational, leads soon to serious difficulties. Prüfer's

theory exorcises these difficulties by not raising them. That is, his proofs are cast ab initio in a mould which is independent of the properties of the rational numbers quâ rational numbers, and in particular the unique decomposition of a rational integer into rational primes is nowhere assumed in the proofs of the fundamental theorems. The theory of the rational integers is therefore included, not presupposed as customary, in the new theory.

The rational unique factorization theorem is here replaced by another, due to Dedekind, and this substitute theorem does not imply, nor is it implied by, the rational theorem. It is this: if $\alpha_j (j=1, \cdots k)$ belong to an algebraic number field, then in the same field there exist numbers $\beta_j (j=1, \cdots, k)$ such that each product $\alpha_i\beta_j (i, j=1, \cdots, k)$ is an integer and $\sum \alpha_i\beta_i = 1$ $(i=1, \cdots, k)$. Having avoided by this means the peculiarities of the rational field in his proofs, Prüfer attains a theory which includes that of the rational integers as a special case, and moreover it is clear that, no particular field being given precedence over any other in the demonstrations, the theory of relative fields is no longer necessary for complete generality.

Prüfer's theory has been recast by v. Neumann, who also has removed certain restrictions (concerning "infinite" numbers and divisors of zero) of the original theory. To give some idea of this latest development of arithmetic we shall follow v. Neumann's exposition so far as may be done with the concepts of Dedekind's theory already sketched.

14. In Prüfer's theory are defined certain "ideal numbers" which accomplish unique factorization, as in the Dedekind theory, and which, unlike the ideals of that theory, form a ring. Ideals (Dedekind) and ideal numbers (Prüfer) are distinct things; we shall use these terms in full, so that "ideal" is not an abbreviation for "ideal number." To each Dedekind ideal corresponds (in a technical sense) in Prüfer's theory a class of ideal numbers, and there exist ideal numbers corresponding to no Dedekind ideal. Prüfer's theory is therefore a true generalization of Dedekind's.

An ideal number to which corresponds no Dedekind ideal is called infinite; the rest, except zero and its divisors, are called finite. Finite ideal numbers, as in the Dedekind theory, have a unique factorization theorem. Prüfer introduces his ideal numbers as 'ideal solutions' of infinite systems of congruences; v. Neumann reaches results similar to Prüfer's otherwise and he obtains a multiplicative decomposition (unique factorization) for all ideal numbers, including the finite, the infinite, zero and the divisors of zero. The finite ideal numbers justify their name thus: a finite ideal number differs from the other varieties only in that it is the product of a finite number of factors. When an infinity of factors are required in the decomposition the ideal number is called infinite; if at least one prime factor is repeated an infinite number of times the ideal number is a divisor of zero; if all prime factors enter, each an infinity of times, the ideal number is zero.

One stumbling block in the terminology must be avoided. "Real number" is used always in contradistinction to "ideal number." Thus in this theory a real number is in general complex. The real numbers of Prüfer and v. Neumann are the elements (algebraic numbers) of any given algebraic number field of degree n. The paradox then is one of nomenclature, not of fact. Dedekind's theory is presupposed, at least in v. Neumann's presentation.

We proceed to the explanation of v. Neumann's excessively abstract ideal numbers. The following definitions and theorems are as beautiful an example of the arithmetical formulation of a theory as any in the literature. They can be grasped on their own merits independently of their intrinsic significance in relation to algebraic numbers.

15. Rational numbers are denoted by $m, n, \cdots$; real numbers by $\alpha, \beta, \cdots, \xi, \eta, \cdots$; Dedekind prime ideals by $\mathfrak{p}, \mathfrak{q}, \cdots$; sequences of real numbers by $R, S, \cdots$; ideal numbers by $\mathfrak{A}, \mathfrak{B}, \cdots$. The previous notation $* \mid \dagger$ for "$*$ divides $\dagger$" is retained; "divides" can in each case be read "includes" or "contains" in the sense of class inclusion as in mathematical logic. The case of $\alpha \mid \beta$, an apparent exception, can be referred to $(\alpha) \mid (\beta)$, (α), (β) principal ideals.

Let α be a real number, m a rational integer, $\mathfrak{p}$ a prime ideal. Then α is said to have the factor $\mathfrak{p}^m$ when there exists a real (algebraic in the given field), integer ξ such that $\mathfrak{p}^m \mid \alpha\xi$.

A sequence $R \equiv [\alpha_1, \alpha_2, \cdots]$ of real numbers is called fundamental when for every $\mathfrak{p}$, m almost all ($\equiv$ all with a finite number of exceptions) differences $\alpha_r - \alpha_{r+1}$ have the factor $\mathfrak{p}^m$.

Any two sequences $R \equiv [\alpha_1, \alpha_2, \cdots]$, $S \equiv [\beta_1, \beta_2, \cdots]$ of real numbers are called equivalent, $R \sim S$, when for each $\mathfrak{p}$, m, almost always, $\mathfrak{p}^m \mid \alpha_r - \beta_r$ (viz., with a finite number of exceptions the differences $\alpha_r - \beta_r (r = 1, 2, \cdots)$ are elements of the mth power of the prime ideal $\mathfrak{p}$).

Hence equivalence of sequences of real numbers is abstractly identical with equality, being reflexive, symmetric, and transitive. Moreover the set of all sequences of real numbers falls into mutually exclusive classes such that all elements (sequences) of a given class are equivalent and no two elements of different classes are equivalent. Further, if $R \sim S$, then both or neither of R, S are fundamental. Hence all or none of the sequences in a particular class are fundamental. Thus classes of sequences fall into two classes, fundamental and non-fundamental; each sequence in a fundamental class is fundamental; no sequence in a non-fundamental class is fundamental.

A fundamental class is now called an ideal number.

If R is an element of the ideal number $\mathfrak{A}$, then $\mathfrak{A}$, R are said to appertain to one another, and this is written $\mathfrak{A} \sim R$. Each fundamental R appertains to precisely one ideal number; to each ideal number appertain an infinity of R's.

The relation $\mathfrak{A}\sim R$ has the abstract properties of equality; thus $\mathfrak{A}\sim R$, $R\sim S$ imply $\mathfrak{A}\sim S$; $\mathfrak{A}\sim R$, $\mathfrak{A}\sim S$ imply $R\sim S$; $\mathfrak{A}\sim R$, $\mathfrak{B}\sim R$ imply $\mathfrak{A}\sim\mathfrak{B}$.

Fundamental sequences $R\equiv[\alpha_1, \alpha_2, \cdots]$, $S\equiv[\beta_1, \beta_2, \cdots]$, $\cdots$, form a ring in which addition $R+S$, subtraction $R-S$, multiplication RS are the operations yielding respectively the sequences (easily shown to be fundamental)

$$[\alpha_1+\beta_1, \alpha_2+\beta_2, \cdots], \quad [\alpha_1-\beta_1, \alpha_2-\beta_2, \cdots], \quad [\alpha_1\beta_1, \alpha_2\beta_2, \cdots].$$

Hence $R'\sim R''$, $S'\sim S''$ together imply

$$R'\pm S'\sim R''\pm S'', \qquad R'S'\sim R''S''.$$

Let $\mathfrak{A}\sim R$, $\mathfrak{B}\sim S$($\mathfrak{A}$, $\mathfrak{B}$ ideal numbers). Then the ideal numbers appertaining to $R\pm S$, RS respectively depend only upon $\mathfrak{A}$, $\mathfrak{B}$ and not upon R, S.

Hence we define the sum $\mathfrak{A}+\mathfrak{B}$, the difference $\mathfrak{A}-\mathfrak{B}$, the product $\mathfrak{A}\mathfrak{B}$ of the ideal numbers $\mathfrak{A}$, $\mathfrak{B}$ as the ideal numbers appertaining respectively to $R+S$, $R-S$, RS.

Under this definition ideal numbers form a ring, since obviously addition, subtraction and multiplication of ideal numbers as just defined obey the commutative, associative and distributive laws and since, moreover, subtraction as defined is the unique inverse of addition.

An integral ideal number is one to which appertains at least one fundamental sequence all of whose elements are algebraic integers. Hence, immediately, if $\mathfrak{A}$, $\mathfrak{B}$ are integral ideal numbers so also are $\mathfrak{A}\pm\mathfrak{B}$, $\mathfrak{A}\mathfrak{B}$.

Each sequence of the type $[\alpha, \alpha, \cdots]$ is fundamental; the ideal number appertaining to $[\alpha, \alpha, \cdots]$ is written[1] α'.

Hence we have

$$\alpha'+\beta'=(\alpha+\beta)', \ \alpha'-\beta'=(\alpha-\beta)', \ \alpha'\beta'=(\alpha\beta)'$$

where $(\alpha+\beta)'$ is the ideal number appertaining to $\alpha+\beta$, and similarly for the others; also

$$\mathfrak{A}+0'=\mathfrak{A}-0'=\mathfrak{A}, \qquad \mathfrak{A}-\mathfrak{A}=0',$$
$$\mathfrak{A}1'=\mathfrak{A}, \qquad \mathfrak{A}0'=0'.$$

Division of ideal numbers is defined as the inverse, in the arithmetical sense of multiplication. Thus, if there exists an integral ideal number $\mathfrak{G}$ such that $\mathfrak{B}\sim\mathfrak{A}\mathfrak{G}$, then by definition $\mathfrak{A}\mid\mathfrak{B}$, and this division has the abstract properties of arithmetical division, so that in particular

$$\mathfrak{A}\mid\mathfrak{B}.\ \mathfrak{B}\mid\mathfrak{C}:\supset:\mathfrak{A}\mid\mathfrak{C};$$
$$\mathfrak{A}\mid\mathfrak{B}.\ \mathfrak{A}\mid\mathfrak{C}:\supset:\mathfrak{A}\mid\mathfrak{B}\pm\mathfrak{C};$$
$$\mathfrak{A}\mid 0', \qquad \mathfrak{A}\mid\mathfrak{A},$$

in which $\supset$ is the customary symbol for "implies," and, for example, the first states that if $\mathfrak{A}$ divides $\mathfrak{B}$ and $\mathfrak{B}$ divides $\mathfrak{C}$, then $\mathfrak{A}$ divides $\mathfrak{C}$.

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[1] I have here changed v. Neumann's bars to accents to suit machine printing.

The connecting link with common division is the following easily seen theorem,

$$\alpha \mid \beta : \equiv : \alpha' \mid \beta',$$

that is, each of these division relations implies the other.

Recalling the definition of the relation "α has the factor $\mathfrak{p}^m$" which was given near the beginning of this section, we can now close this chain of definitions and theorems as follows. The ideal number $\mathfrak{A}$ is said to have the factor $\mathfrak{p}^m$ when, for each fundamental sequence $R \equiv [\alpha_1, \alpha_2, \cdots]$ appertaining to $\mathfrak{A}$, almost all α_r have the factor $\mathfrak{p}^m$. Hence if $\mathfrak{A}$ has the factor $\mathfrak{p}^m$ it also has the factor $\mathfrak{p}^n$, $n \leqq m$; if also $\mathfrak{A} \mid \mathfrak{B}$, then $\mathfrak{B}$ has the factor $\mathfrak{p}^m$, and then each of $\mathfrak{A} \pm \mathfrak{B}$ has the factor $\mathfrak{p}^m$; while if $\mathfrak{A}$, $\mathfrak{B}$ have the respective factors $\mathfrak{p}^m$, $\mathfrak{p}^n$, then $\mathfrak{AB}$ has the factor $\mathfrak{p}^{m+n}$. Finally, either both or neither of α', α have the factor $\mathfrak{p}^m$; the equations $\alpha' = \beta'$ and $\alpha = \beta$ are formally equivalent, and in all operations which have been defined for both real and ideal numbers (addition, subtraction, multiplication, divisibility, having a factor $\mathfrak{p}^m$), the real numbers α and the ideal numbers α' play precisely the same parts. Hence the theories of real and of ideal numbers are abstractly identical up to the point considered, and the structure of the ideal theory is abstractly identical with that of rational arithmetic.

For the ideal numbers above described, v. Neumann obtains the results summarized in § 14. He also finds much more, and his further theory, like that of Prüfer, has close affiliations with Hensel's theory of p-adic numbers. It also, in its definitions concerning limits, is allied to Kurschak's work. Thus the scope of the Prüfer–v. Neumann theory is wide indeed. Its further development will be eagerly awaited. At the present time a detailed application to some number field of degree greater than 2 would be welcome, as only by concrete application can the power of a theory of algebraic numbers in obtaining new results be appraised.

16. Looking back over the three specimen theories of algebraic numbers selected to illustrate some of the features of such theories, and having in mind numerous other theories whose aims are the same or closely similar to those of the three chosen, we see the need of several definite projects, all of which are well within reach of our existing technique. Of these things to be done we need mention only a few; others will occur to anyone who tries to find his own way.

First, as a general enquiry, it may be asked why have some theories of algebraic numbers survived and others perished? What are the defects, if any, which have embalmed certain theories, for example those of Zolotareff and Sochocki, as beautiful things but dead, in the short notices of the *Fortschritte*, while others still masquerade in at least the semblance of life? This enquiry however may be only of antiquarian interest.

More vitally, it would not be a waste of time to exhibit each of these theories, dead, living and in process of being born, in purely abstract form, so that we may see at a glance precisely what the theory is as an object, not of art, but of pure mathematics. Until this be done it seems unlikely that we shall be able to say in what significant way one theory differs from another.

Having abstracted the several theories we may classify them and isolate the part common to all as abstract arithmetic. Rational arithmetic falls naturally into two main divisions, additive and multiplicative. It is perhaps too much to expect that any considerable part of both divisions shall pass over into the abstraction common to several theories.

With the relational aspects of the theories before us, we may modify the abstract structure of the postulates and develop the theories of the resulting systems. This has been done, for example, in Dickson's theory of the arithmetic of a rational algebra. In other examples some curious possibilities emerge. Thus it appears that we may have sets of integral elements for which division is not unique but which posses a unique multiplicative decomposition into indecomposable elements of the set.

Incidentally the postulational treatment of the whole of existing arithmetic is a desideratum. The like discussion of projective and other kinds of geometry yielded unexpectedly interesting results; the same may prove to be the case for arithmetic. Sets of postulates, say for the G. C. D., the L. C. M., arithmetical divisibility (as opposed to algebraic), integral elements, sufficiently broad to include the known instances, do not seem to have been attempted.

How far is it possible to go in arithmetic without using order relations? Again, exactly what is arithmetic and what part of any given theory (including the theories of geometry) is abstractly arithmetical in structure? Finally there is the problem of generalizing each definite abstraction achieved and producing interpretations of the generalizations.

If one guess may be permitted, it is this: a thoroughgoing examination of the pure mathematics underlying the numerous theories of algebraic number fields of the past half century will show that no one of these theories has gone much if at all deeper than the profound work of Richard Dedekind.

17. In conclusion we recall the famous aphorism of Kronecker: "God made the integers; all the rest is the work of man." History does not record whether Kronecker intended the second part of this remarkable theorem as a boast on behalf of man's superior ingenuity or a lament over his prolific perversity. It may be taken either way. Looking soberly over the welter of arithmetic theories that have jostled their way into existence after Gauss' initial improvement over $1, 2, 3, \cdots$, a pessimistic critic might mistake the theory of numbers for a branch of physics.

NOTE BY THE EDITOR: In a later issue of this Monthly there will be published a short list of readings in connection with the topics mentioned in Professor Bell's paper.

THE VALUE OF MATHEMATICAL MODELS AND FIGURES[1]

By ARNOLD EMCH, University of Illinois

1. The reason for the choice of the title for this paper is not to put up a defense for the use of models and figures in mathematical instruction. The value of the heuristic element, especially in elementary mathematical teaching, has been and is presumably recognized by the majority of teachers of mathematics. But even in more advanced fields and certain domains of mathematical research the establishment of the constructive features of a mathematical theory may sometimes be greatly aided and illuminated by appropriate graphs, diagrams and models.

Not infrequently it happens that, after the actual construction of a figure or of a model, a close examination of the finished product reveals or suggests the existence of new properties of the form investigated which were not anticipated before the construction.

Another important factor in the construction of figures and models lies in its strengthening of the geometrical imagination and of mathematical intuition in general.

The use of physical images of certain mathematical forms and relations, moreover, is an incomparable aid in fortifying mathematical memory. The acquisition of a concrete picture of a certain theory or of certain forms is a powerful means of mathematical orientation. As a matter of fact, mathematical symbols and formulas must be classed in this category of concrete pictures of mathematical relations. Geometrical figures in the plane and in space (models) serve in a general way the same purpose.

The ability to visualize mathematical truth, gained by training in constructive methods, together with a keen logical reasoning faculty and a strong intuition make a very powerful mathematical combination. These abilities are not always found combined among the mathematicians.

Steiner is a conspicuous example of a visualizing and intuitional genius lacking the critical power of analysis even as it existed at his time. Much of his early work, n its initial stage, was accompanied by marvelously drawn figures. Later, after long practice, when his mental visualizing power had become very strong, he did not find it necessary to make use of figures any more. This personal custom of Steiner was unfortunately carried to an extreme by some of his disciples and successors, like R. Sturm, who in his magnificent "Lehre von den geometrischen Verwandtschaften" has not drawn a single figure.

Klein, among the modern mathematicians of note, may be mentioned as one who combines both intuitional and critological power.

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[1] This is an abstract of a paper, with screen projections, presented to the Mathematical Association of America at Columbus, Ohio, September 7, 1926.

It seems to me that the rejection of either the standpoint of utmost rigor or that of intuitional methods from mathematics in general must lead in the end to one-sidedness or sterility. A happy combination of both seems to be the most fruitful. No historic instance will illustrate this better than that of Riemannian function theory and Riemannian geometry, which have opened up entirely new mathematical vistas, which the Berlin school under the leadership of Weierstrass was unable to foresee.

It must be understood, of course, that in recommending the use of figures and models, these must be merely considered as useful auxiliaries, not as ends

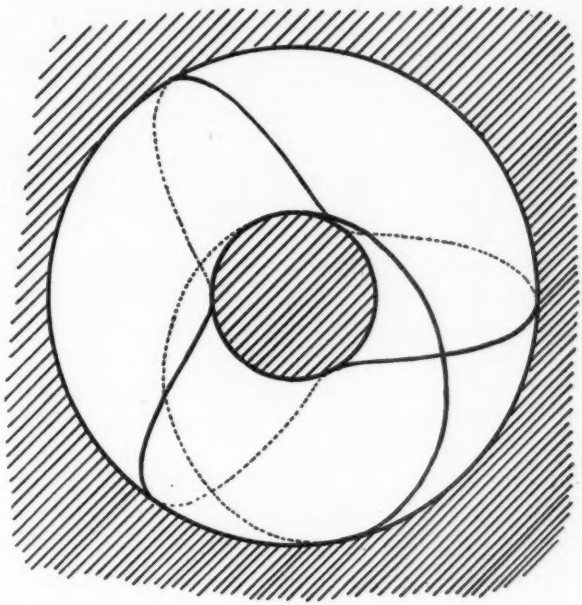

FIG. 1

in themselves, although they may often prove to be of great value in bridging over to applied mathematics.

I shall now speak specifically of some examples which may help to illustrate and clarify some of the ideas expressed above, in order to point out the possibilities of mathematical visualization for mathematical instruction by such means.

2. The plane sections of a torus form a particularly interesting class of bicircular quartics, i. e. quartics with the bicircular points at infinity as double points. Their self inverse (anallagmatic) properties are well known. On account of their extremely simple construction and representation they may

easily be visualized and classified as to their genus. A generic plane cuts the torus in a bicircular quartic of genus 1; a simple tangent plane, in a rational quartic, a Bernoulli lemniscate for example, with the point of tangency as the double point. When the intersecting plane is doubly tangent, the quartic, having now four double points, two real and two imaginary, must necessarily degenerate into two conics which, passing through the circular points, consist of two circles intersecting at the two points of tangency. This is beautifully illustrated by a model which shows the additional property that those circles are loxodromics, i. e., lines which intersect the parallels, or meridians, of the torus at a constant angle. This opens the question of loxodromics on a torus in general, which was answered by the author in connection with an application of elliptic functions. When the radius of the core-circle is R, the radius of the rotating, generating circle is r, and a closed loxodromic L_1 winds p_1 times around the axis of the torus and q_1 times around the core-circle of the torus, then the orthogonal loxodromic closes also after p_2 and q_2 similar windings when satisfying the relation $(p_1/q_1)(p_2/q_2)=(R^2-r^2)/r^2$ with R^2/r^2 rational.

Figure 1 illustrates the case $p_1=q_1=1$ of a loxodromic circle ($R=2r$). For the orthogonal loxodromic, $p_2=1$ and $q_2=3$ in this case. The projection is a sextic.[1] In case of closure the loxodromics are algebraic curves.

As another example of effective visualization may be mentioned the space-sextic curve of genus 4, which is obtained as the intersection of a quadric and cubic surface. Projecting the sextic from a generic point upon a generic plane a plane sextic with 6 double points is obtained. The 6 apparent double points of the space sextic are beautifully seen in the model—Figure 2. Moreover the possibilities of real tri-tangent planes of the space sextic are easily seen from the same model. It is not difficult to construct the space sextic of genus four with the maximum number of ovals, namely five,[2] from which may be enumerated the 120 tri-tangent planes.

3. Group theory is a field in which visualization by means of constructive geometry has not been used extensively. And yet modern geometric research in many of its ramifications is closely associated with certain groups. It is true that the regular triangle, the square, the regular polygons in general, and the regular polyhedrons offer beautiful geometric illustrations of certain important finite groups; likewise the well known regular divisions of the complex plane, by circular arcs, which are associated with the groups of automorphic functions and the movements of non-euclidean geometry.

[1] The result contained in the preceding formula, referred to by E. Pompiani as Emch's theorem, was generalized in a memoir by Pompiani on *Rappresentazione grafica delle ciclidi di Dupin e delle loro lossodromiche*, Reale Istituto Lombardo di Scienze e Lettere (3) vol. 21 (1915), pp. 205–242. The author's own investigations on loxodromics were published in this Monthly, vol. 6 (1899), pp. 136–139 and vol. 9 (1902), pp. 277–280.

[2] Hilbert, Mathematische Annalen, vol. 38, 1891, p. 122.

As a new example I mention the model[1] for the symmetric group of order 24 on four letters. If the four letters represent four real numbers, then their 24 permutations represent the substitutions of the symmetric group G_{24} of order 24 and at the same time 24 points in projective 3-space. These 24 points lie on a quadric and by sixes on 16 conics, whose planes by fours pass through four lines of a plane quadrilateral. They lie by twos on 72 lines which pass by twelves through 6 vertices of said quadrilateral, thus belonging to 6 involutions on the

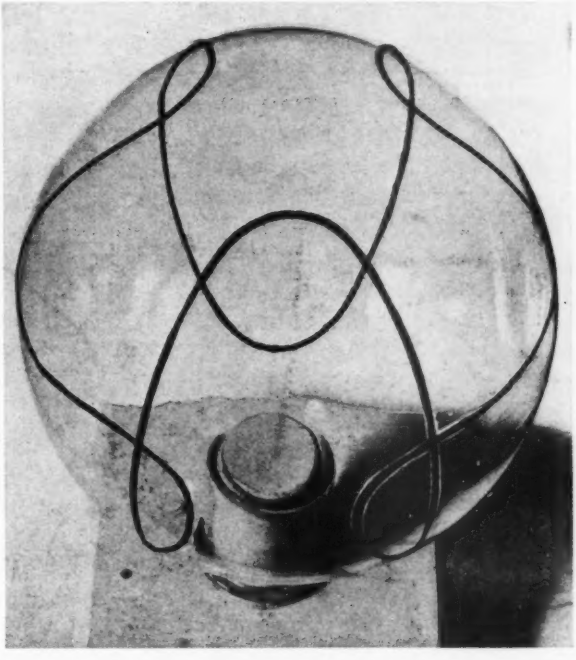

FIG. 2. On the models the lines run smoothly and evenly. Refraction and reflexion of light mar the effect on the photographs and are the cause of irregularities.

quadric. The model reveals the fact, which may be verified, that in certain groups, their joins meet in points outside of the points mentioned before. This would hardly be anticipated without the model.[1]

4. I hope that these examples from various mathematical realms, whose number might be increased indefinitely, will be impressive enough to show that mathematical visualization even in more advanced mathematical instruction and certain lines of research may often serve as a useful auxiliary to strengthen mathematical intuition and initiative.

[1] *Some geometric applications of symmetric substitution groups.* American Journal of Mathematics, vol. 45 (1923), pp. 192–207.

ON THREE INTERESTING TERMS RELATING TO AREA

By SOLOMON GANDZ, Rabbi Isaac Elchanan Theological Seminary, New York.

In the early texts on geometry there are three terms relating to area which have unusual etymological interest. They show, as is not infrequently the case in scientific literature, an interrelation among ancient peoples that is significant in the history of primitive culture. Since these terms do not seem to have heretofore been discussed from the standpoint of comparative philology, and since they have great mathematical interest, it is proposed to set forth the problem briefly as it appears in the light of present knowledge.

The first of these terms is *meshîḥah*, as it appears in its earliest form, at which time it seems to have been common to all Semitic languages. The earliest Arabic geometry, so far as known is the so-called *Bab-al-Misâḥah*, as it appears in al-Khowârizmî's[1] algebra[2] (c. 830). The word *Misâḥah* has three meanings: (1) mensuration,—the act and process of measuring: (2) area or superficial content,—the result obtained by measuring; and (3) geometry, —the art of measuring, including the rules for finding areas and volumes; all three of these meanings being recognized by al-Khowârizmî himself.

In the title *Bâb al-Misâḥah* the word means geometry, this particular chapter of his algebra furnishing rules for ascertaining areas and volumes. In the first sentence of the chapter,[3] however, it is used to mean area. The sentence, which defines the latter term, should be translated: "Know that *one in one* certainly means area and its meaning is one cubit in[4] one cubit." The term is used later[5] with the same meaning, but on the same page[6] it is used to denote mensuration in the sense of the process of measuring. The sentence should be translated as follows: "And truly the process of measuring this is that you shall measure its surface[7] and ascertain its area. And this, multiplied by the depth, gives the volume."[8] Rosen, in his translation, failed to distinguish between these two meanings, and hence a confusion naturally resulted. He gave "mensuration" as the title and then translated the first sentence as follows: "Know that the meaning of the expression 'one by one' is mensuration,"—which has no sense.

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[1] Mohammed ibn Musa al-Khowârizmî.

[2] Rosen ed., London, 1831; Arabic text, pp. 50–64; English text, pp. 70–85.

[3] Arabic text, p. 50, lines 11–12.

[4] This use of "in" to denote multiplication is the origin of our use of "into" in the expression "$a+b$ into $a-b$." Compare Smith, *History of Mathematics*, II, 102. See also the note at the end of this paper.

[5] Arabic text, p. 53, lines 10, 18.

[6] Lines 7, 8.

[7] That is, of the base.

[8] Rosen (p. 74) translates this: "You must calculate it by ascertaining at first the area of its basis. This, multiplied by the height, gives the bulk of the body." This gives correctly the sense of the passage, but does not give the precise meaning of the terms *misâḥah* and *saṭch*.

This first sentence, with the definition of area, was misunderstood not only by Rosen but also by Aristide Marre in his French translation[1] and by Ruska in the German.[2] It is not due to al-Khowârizmî but was evidently taken from a much older work,—the Hebrew geometry called the *Mishnath ha-Middoth*.[3] In this, however, the usual term for area is *Meshîḥah*, this being also used for volume.[4] As a term for area it appears in the form *meshîḥah* in the *Liber Embadorum* of Abraham Savasorda (c. 1100), which bears the Hebrew title *Ḥibbûr ha-Meshîḥah weha-Thishboreth* (*The book of the area and the superficial content*).[5] In the Talmudic and Syriac literature the word *meshîḥah* is also well-known, although there it has the meaning of a surveyor's rope or measuring cord. In each of these languages *mashôḥa* means a surveyor or a geometer.[6] The root *meshaḥ*, *mashaḥ*, or also *mataḥ* means "to extend," "to stretch," "to measure," and appears in the Aramaic and Syriac translations of the Bible for the Hebrew *madad*. It should not be assumed, however, that *mashaḥ* is a later form, since there is reason to believe that the biblical word *madad* is the later Hebrew word for the much older common Semitic root *mashaḥ*, a form that was in common use in Assyrian literature.[7] The word *meshîḥah* as it appears in the *Mishnath ha-Middoth* is undoubtedly used as a well-known ancient scientific term for which the definition is given. The author of the work seems proud and anxious to write in the classic Hebrew style, and twice he tries to introduce the biblical word *medîdah* instead of *meshîḥah*.[8] This he has to give up, however; the old habit is victorious and the ancient term *meshîḥah* holds place throughout the rest of the book as the only word for area. It seems clear, therefore, that we may safely assume the existence of an ancient and well-known Semitic term for area namely *meshîḥtu*, *meshîḥah*, or *misâḥah*. The earth measurers (geometers) were called *mashîḥanu* or *mashôḥa*, in Arabic *massâḥ*; they were the measurers of area, the ones who handled the measuring cords (*meshîḥah*).

[1] *Annali di Matematica pura ed applicata*, VII, 271.

[2] *Zur ältesten Arabischen Algebra*, p. 103, note.

[3] The date is very uncertain, as stated by Professor Smith, *History*, I, 174; but the writer is confident that it belongs early in the Christian era. The definition appears in I, 6.

[4] See I, 2, 3, 4, 5, 6, 7, etc., for the area, and II, 6, 7, 9 for the volume (*meshṭḥath ha-gûf*, or *ha-'ammûd*).

[5] The word *Thishboreth* is discussed later.

[6] See the Talmudic dictionaries of Levy, Jastrow and Kohut, and the *Thesaurus Syriacus* (II, 2235 seq.) of Payne Smith.

[7] See the Assyrian dictionaries of Delitzsch (pp. 430, 431) and Muss -Arnold (pp. 600, 601), which relate *mashaḫu* (to measure, to measure a field) *meshiḫu* (measure in general, also measure of a quantity of grain or dates), *meshiḫtu* (measure, extent of ground, field), and *mashîḫanu* (land surveyor). In *meshiḫtu* we probably have the old Assyrian word for area.

[8] See the *Mishnath ha-Middoth*, I, 1, 6, where *medîdah* is properly translated "area" (not understood by Schapira). The passage I, 1 should be translated: "There are four ways to grasp (ascertain) the area"; and (I, 6), "and the *medîdah* (area) contains four times the square unit."

We also find traces of the measuring cord in early Egypt, used in connection with length and with the square unit employed in land measure. The *khet*, meaning the reel of cord, was 100 cubits long, and its square was the unit of area, the Egyptian *setat*, in Greek the *ároura* (ἄρουρα).[1] The Greek writer Democritus (c. 400 B.C.) refers to the *harpedonaptae* (ἁρπεδονάπται), rope stretchers,[2] possibly analogous to the Hindu *śulvasūtra* and to the Semitic *mashôḥa*. The Akkadeans, the ancient predecessors of the Assyrians, also used *ashlu* (cord) and *śubban* (half cord) as terms for measure.[3]

We therefore find an ancient term adopted by all Semitic writers; this term is analogous to Greek, Egyptian, and Sanskrit terms, all relating to surveying, and the parent in idea though not etymologically of the Greek word which gave us the term "geometry."

The second of the terms under consideration is the Greek word *chroia* (χροιά, χρῶμα; chroia, chroma), meaning "skin" and "color." It was used by the Pythagoreans,[4] but the date of its first appearance is quite uncertain. It is derived from the word *chrio* (χρίω, χράω, χραίνω, χρώξω), meaning to "rub," "anoint," "paint," "smear," and in general "to slightly touch the surface of a body." Now it is worth while to notice the relation of *chroia*, which became the Greek term for "surface," to the Semitic *meshîḥah*, already mentioned. Besides the meanings stated in the first part of this article it also means, like the Greek word, to "rub," "anoint," or "smear" the skin. Hence we have the Hebrew *mashîaḥ*, *Messiah*, and the Greek *Christos*, the anointed one. The Assyrian language recognized only the geometric meaning (to measure); the biblical Hebrew, only the second one (to anoint); while the Syriac, the Aramaic, the post-biblical Hebrew, and the Arabic used both meanings. From the first root of *mashaḥ* (to measure) there developed in the Semitic languages the term for area, while in the Greek language this came from *chroia*, corresponding to the second root of *mashaḥ* (to anoint). This suggests a possible connection between the two roots,—to touch the surface lightly either with the hand or with a measuring cord. On the other hand the Greeks may have received the word for area from the Semitic as they are known to have received many others.

This connection of geometric terms in different languages is also seen in the case of the late Greek expression τὸ ἐμβαδόν (*embadon*, *embadum*, area, as in the *Liber embadorum*). It comes from the word ἐμβαίνω (*embaino*, ἐμβαδίζω, *embadizo*, to step upon, walk apace, and then to measure with the feet

[1] T. E. Peet, *Rhind Mathematical Papyrus*, London, 1923, pp. 24, 25; F. L. Griffith, *Proceedings of the Society of Biblical Archaeology*, XVI, 236, seq.

[2] See T. L. Heath, *Greek Mathematics*, I, 121; Smith, *History*, I, 81; II, 288; Peet, *Rhind Papyrus*, p. 32. The Greek word is from ἁρπεδόνη, rope or cord, and ἅπτω, to seize or handle.

[3] *Revue d' Assyriologie*, XVII, 133. Compare also the English "chain" as unit of measure.

[4] Smith, *History*, II, 276.

or by pacing).[1] Now the Arabic *masaḥa* (Semitic *mashaḥ*) means not only to measure and to survey a field, but also to walk apace, as in the Greek. The verse in Isaiah XL, 12, "Who hath measured the waters in the hollow of His hand and meted out heaven with the span," should, according to Rashi (c. 1100), read "Who hath measured the waters with the hollow of His sole or with His steps."[2]

We therefore have an interesting connection relating to the geometric *superficies*, surface, measure, area, skin, rope stretchers, *śulvasutra* (?), *harpedonaptae*, *chroia*, and *meshîḥah*.

The third of the curious names for area is *taksîr* or *thishboreth*, a word meaning "to break into pieces" and hence "fractions." After using *misâḥah* for area, as already stated, al-Khowârizmî changed to the word *taksîr* and used this throughout the chapter, with only a few exceptions. These exceptions were where he wrote *misâḥah* because he was in the immediate context using *taksîr* for volume.[3]

The question now arises as to the origin and meaning of *taksîr*. Thus far we have found area represented only by terms relating to surface, appearance, color, extent, measure, and the like,[4] it being quite clear why they were chosen. It is, however, not at first sight apparent why *taksîr* was chosen,—a word meaning fractions, or to break in pieces. Naturally we turn to Hindu literature, the source of so many ideas of al-Khowârizmî, and here we find a similar term, *cuṭṭaca* or *cuṭṭa*, meaning a pulverizer or a grinding, pulverizing multiplier.[5] This word, however, is used both for algebraic computations in general and for a special algebraic process,—namely to find a quantity such that when a given number is multiplied by it and the product is added to another given number, the sum is exactly divisible by a given

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[1] The same idea occurs in the Latin *podismus*, Greek ὁ ποδισμός measuring by the feet, area. (Tropfke, IV, 2nd ed., pp. 37, 126–127.)

[2] Compare also Rashî on Numbers XXII, 42. Rashî's opinion receives support in the Chaldean use of the word *sha'ala* for hollow of the sole, step. See the dictionaries of Levy and of Jastrow.

[3] See p. 53, lines 10, 18. The name *taksîr* is used regularly for area on pp. 51 (lines 7, 9, and at the bottom), 52 (lines 3, 4, 15), 53 (lines 3, 7), 54 (lines 8–10), 57 (lines 14, 16), 55 (lines 8–14), 56 (lines 3–6, 12), etc. For volume the same word is used on pp. 53 (5, 8, 10), 62 (line 11), and 63 (line 6).

This use of *taksîr* is also given in the dictionaries of Lane (p. 2612^c) and Dozy (II, 465). The latter quotes many geographical writers who use the term. It often occurs in the Arabic writings of the Hebrew philosopher Moses ben Maimon (c. 1200); see Friedlaender, *Lexicon to Maimonides*, under *kasara*; in addition to this see the manuscript commentary of Maimonides on the *Mishna Erubin*, I, 5, and II, 5, now in the Jewish Theological Seminary, New York City.

The new term *taksîr* did not, however, entirely crowd out the old *misâḥah*, and so we find Omar Khayyam (Khayyâmî, c. 1100) and Beḥâ Eddîn (c. 1600) using only the term *misâḥah* and *saṭḥ* (roof, surface). See Omar Khayyam, ed. Woepke, pp. 9 (lines 10, 12, 18), 27 (line 3), 29 (line 5), and the chapter on geometry in Beḥâ Eddîn's *Kholâsat al-Ḥisâb*.

[4] Smith, *History*, II, 276; Tropfke, IV, 2nd ed., pp. 37, 126–127.

[5] H. T. Colebrooke, *Algebra · · · · from the Sanscrit*, London, 1817, pp. 112, 113 seq., and 325 seq.

divisor.[1] It therefore appears that the term does not assist us with respect to the word *taksîr*.

Turning then to the Hindu terms for area, we find in the writings of Bhāskara (c. 1150) and Brahmagupta (c. 628) the words *phala*, *gaṅita*, *cshêtra-phala*, and *sama-cóshta-miti*, which the commentators interpret to mean the "measure of like compartments, or number of equal squares of the same denomination (as cubit, fathom, finger, etc.) in which the dimensions of the side is given: i. e., the area or superficial content." Another explanation reads, "The area of a figure is represented by little square compartments formed by as many lines as are the numbers of the upright and side."[2] The related term *g'hana-phala* is used for the solid content, apparently meaning a number of equal cubes of the same denomination.[3] This suggests the idea that the term *taksîr* had the origin in the idea of breaking the whole area into equal square parts or units, thus obtaining an exact measure of superficial content.

This idea is corroborated by the medieval Hebrew use of this very term. The Hebrew name of the *Liber Embadorum* of Savasorda is *Ḥibbûr ha-Meshîḥah weha-Thishboreth*, *Meshiḥah* meaning "area," as already stated, and *Thishboreth* (breaking) being both a translation and a formal imitation of the Arabic *taksîr*, and very likely being first used in this work[4] (c. 1100). Although both *Meshiḥah* and *Thishboreth* are used, in the text itself the former gives place to the latter, a procedure already followed by al-Khowârizmî, as we have seen. Since that time *Thishboreth* and *Shibbûr* have been the current Hebrew terms for both area and geometry,[5] but none of this assists us in our understanding

[1] *Ibid.*, p. 113, note 1.

The best explanation of the term *cuṭṭaca* seems to be that given by Al-Bīrûnî, (c. 1030) in his *India*, ed Sachan, Arabic text, p. 74, lines 8-10, Engl. text, I, p. 155: "19. On *Kuṭṭaka*, i. e. the pounding of a thing. The pounding of oil-producing substances is here compared with the *most minute and detailed research*. This chapter treats of algebra and related subjects." The same association of ideas is to be found in the Hebrew form *diqdûq*. It means "pulverising," or "pounding of oil-producing substances," and is used as a term for the science of grammar with its minute research. *Diqdûq haberîm*, "the pounding of the fellow-scholars" is used for the dielectic discussions on Talmudic law.

[2] *Ibid.*, p. 69, note 6; p. 296, note 2.

[3] *Ibid.*, p. 88, note 2.

[4] Abraham bar Chiia Savasorda (c. 1070–c. 1136). As is well known the name Savasorda came from the Arabic *Sâḥib al-Shorta*, "Chief of the Guards." It seems first to have been used by Plato of Tivoli (c. 1120), see Smith, *History*, I, 206. Professor Guttman, introduction to the *Ḥibbûr ha-Meshihah*, p. XIII, note 3; and Professor Wolfsohn, *Jubilee Volume of the Hebrew Union College*, Cincinnati, p. 301, are both mistaken in their derivation of *Thishboreth* from the Arabic *shibr*. This would contradict all rules of Semitic philology, as well as Savasorda's own statement (*Ḥibbûr ha-Meshihah*, p. 23) that it is a translation from the Arabic *taksîr*.

[5] See the Hebrew astronomy of Savasorda *Ṣûrath ha- Areṣ*, chapter IX; the Hebrew translation of the commentary of Maimonides on the *Misna Erubin*, I, 5; II, 5; IV, 2, in the first edition, Naples, 1492, the later editions erroneously giving *shi'ûr* instead of *shibbûr*. Compare further the astronomy of Isaac Israeli, *Jesod 'Olam*, I, 2, 3, pp. 2b, 3b (2d. ed., Berlin); the astronomy of David Gans, *Neḥmad we-Na'îm*, p. 17; § 31; Luzzato, *Kerem Ḥemed*, II, 70, 76; Steinschneider, *Gesammelte Schriften*, p. 399.

of the key word *taksîr*. We have, however, certain earlier Hebrew works which use the old Aramaic form *thebaryâthâ* (fractions) for area. For example Rabbi Ḥananel of Kairuan (c. 990-1050), in his commentary on *Erubin*, 14b, says: "10 cubits upon 10 cubits are 100 cubits in fractions" (*bi thebaryâthâ*), and the same form appears in his commentary, *Migdal Ḥananel*,[1] in an old Genizah fragment,[2] and in Ibn Ezra's commentary on Hosea, XIV, 6, where he says that "fractions" in geometry are analogous to "roots" in arithmetic.[3] Here then we have the proof that the "fractions" are merely the "square units," called *jadhr* (root, which also corroborates the writer's opinion[4] that the *jadhr* was originally a name for the square unit and not for an abstract root or for a side of one dimension. A similar explanation of the word *shebarîm* (fractions) is also given by Samuel ibn Tibbon in his lexicon of technical terms to Maimonides's *Guide to the Perplexed*.[5]

The clue to the meaning of the term is thus apparent. The ancient mathematicians made a great effort to bring home the idea that square measure is entirely different from linear and that therefore the square unit is not a line but a "fraction," a part of the whole area, and a magnitude of two dimensions. The same reasoning was also responsible for the introduction of the *jadhr*.[6] It is all a part of the age-long efforts to make the immature mind realize that the doubling of the side of a square does not double the area, but quadruples it. The *Mishnath ha-Middoth* (I, 7), for example, uses a figure precisely like the one found in our modern textbooks, and the *Liber Embadorum* of Savasorda does the same.[7] So the Genizah fragment above mentioned says that to find the area, *ḥalqînah bi-Thebaryâthâ* ("divide it into fractions"); Maimonides[8] (c. 1200) says: "*wa-huwa mauḍi'un fî misâḥatihi 'inda at-taksîrî* ("and this is a place whose area is found by breaking it into fractional parts"); and a Hindu commentator[9] remarks: "The area is represented by little square compartments formed by as many lines as are the numbers of the upright and side." The result of this breaking of an area into small fractions is to have a figure resembling the plaiting or twisting of basket work,

[1] Edited by Berliner, p. 38. Berliner confesses (p. XXV) to have written *thishboreth* for *thebaryâthâ*. This is purely a corruption of the text.

[2] The exact date is unknown. It was edited by Professor Louis Ginzberg, *Geonica*, II, 36.

[3] See also his commentary on Numbers, XXXIV, II. Ibn Ezra uses the Hebrew *shebarîm* and the Aramaic *thebaryâthâ* for fractions, and the Hebrew *shorashîm* for roots. *Shoresh* was first used by Ibn Ezra for the Arabic *jadhr*. See this Monthly, vol. 33, p. 262.

[4] This Monthly, vol. 33, p. 263.

[5] Under *thishboreth*. See also Iḫwân as-Safâ, ed. Bombay, p. 37, quoted by Ruska, *loc. cit.*, p. 69, note 1.

[6] See this Monthly, vol. 33, p. 263.

[7] P. 28, 2; Hebrew text p. 23.

[8] Commentary on the *Mishna Erubin*, I, 5. There are three MS. copies of it in the Jewish Theological Seminary, New York City.

[9] Colebrooke, p. 297.

and so we find *Mishnath ha-Middoth* saying (II, 5): *Meşaref ha-orekh bethokh ha-roḥab* ("He who has to measure square figures has to plait, or twist length into breadth"). This is the original meaning of *şaref* or *şoref*, "to plait" or "melt", and later "to multiply."[1]

From the facts here adduced we may fix the approximate dates of introduction of the terms mentioned. The earliest known Hebrew geometry set forth the method of calculating an area by dividing it into "fractions," but did not know the terms *taksîr*, *thebaryâthâ*, or *thishboreth*. Mohammed ibn Musa (c. 800), Rabbi Ḥananel (c. 1000) and the Genizah fragment (c. 900-1000) introduced the terms without any explanation, as if they were well-known at that time. We can trace them to the Hindu terms *phala* and *cshêtra-phala* as given in the works of Brahmagupta (c. 628). Whether this suffices to prove the Hindu origin of the idea must be left to the determination of Sanskrit scholars, and it is hoped that this study will lead them to undertake this definite and important piece of work.[2]

ON THE LEAST MULTIPLE OF AN INTEGER EXPRESSIBLE AS A DEFINITE QUADRATIC FORM

By H. S. VANDIVER, University of Texas

In his *Report on the Theory of Numbers*, H. J. S. Smith[3] discussed methods for the solution of the quadratic congruence $x^2 \equiv D \pmod{P}$ where D and P are integers, D prime to P. After discussing a method due to Gauss,[4] he describes a simplification of it as follows:–

"By a known property of quadratic forms, whenever the congruence $r^2+D \equiv 0 \pmod{P}$ is resoluble, the equation $mP = x^2+Dy^2$ is resoluble for some value of $m < 2(D/3)^{1/2}$, if $D>0$. By assigning therefore to m all values in succession which are inferior to that limit and which satisfy the condition

$$\left(\frac{m}{P}\right) = \left(\frac{P}{D}\right)$$

(these are Legendre's quadratic residue symbols) and then obtaining by Gauss' method all prime representations of the resulting products by the form x^2+Dy^2, we shall have

$$r \equiv \pm x'/y', \quad r \equiv \pm x''/y'', \cdots \pmod{P}$$

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[1] The terms used in multiplication have been mentioned by Professor Smith, *History*, II, 114–116, and 102 seq. The subject is very extensive in relation to the Semitic languages and deserves careful study. To *saref* = "plait" compare *sarîf* = "roof of twisted reeds."

[2] The writer wishes to express his appreciation of the assistance of Professor David Eugene Smith in the preparation of his manuscript.

[3] British Association Report, 1860, Art. 68, p. 120; Collected Mathematical Papers, vol. 1, p.148.

[4] *Disquisitiones Arithmeticae* (1801), Articles 327, 328.

$x', y'; x'', y''$; etc. denoting the different pairs of values of x and y in the equation $mP = x^2 + Dy^2$."

In the first place it is possible to show that this method does not always yield all the roots of the congruence when P is composite. For examine the congruence $u^2 + 11 \equiv 0 \pmod{45}$. It has four incongruent roots $\pm 13, \pm 22$. Following Smith's method we find $1 + 11 \times 2^2 = 45$ and $6^2 + 11 \cdot 3^2 = 3 \cdot 45$. The first relation gives ± 22 as solutions of the proposed congruence, but clearly the second relation, in itself, does not give any roots. Smith's use of the notation $r \equiv \pm x'/y'$, and use of the term prime representation indicate that he meant that x' and y' could always be taken prime to each other, which is contrary to the situation in the above example. Also, in quoting Smith's[1] results the statement "x and y prime to each other" was included by the writer.

The "known property of quadratic forms" referred to by Smith, namely that whenever the congruence $r^2 + D \equiv 0 \pmod{P}$ is resoluble the equation $mP = x^2 + Dy^2$ is resoluble for some value of $m \leqq 2(D/3)^{1/2}$, was proved by Oltramare,[2] for the case P a prime, and by Aubry,[3] for the general case the latter using an unnecessarily complicated method.

In another paper on factoring integers it will be necessary for me to make use of a slightly more specific result which includes Smith's statement.

THEOREM: *If $r^2 + a \equiv 0 \pmod{n}$, where a is positive and prime to $n > 1$, there exists at least one set (x, y) such that*

$$x^2 + ay^2 = kn, \quad k \leqq 2(a/3)^{1/2} \text{ and } ry \equiv \pm x \pmod{n},$$

x and y being positive integers or zero.

The argument we shall employ depends on well known methods in the theory of definite quadratic forms.[4] If $0 < r < n$ and $r^2 + a \equiv 0 \pmod{n}$, a positive, consider the form, $F = (n, r, (r^2 + a)/n)$. The form F is equivalent to a certain reduced form, say (A, B, C), where $C \geqq A \geqq 2 \mid B \mid$. From this we have

$$AC \geqq A^2 \text{ and } B^2 \leqq \tfrac{1}{4}A^2 \text{ whence, } AC - B^2 = a \geqq 3A^2/4 \text{ or } A \leqq 2/(a/3)^{1/2}.$$

If the substitution

$$T: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \alpha\delta - \beta\gamma = 1 \quad \text{converts } (A, B, C) \text{ into } \left(n, r, \frac{r^2 + a}{n}\right),$$

then $$A\alpha^2 + 2B\alpha\gamma + C\gamma^2 = n \quad \text{and} \quad (A\alpha + B\gamma)^2 + a\gamma^2 = An,$$

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[1] Vandiver, Bulletin of the American Mathematical Society, vol. 30, (1924) p. 548, last line.

[2] Journal für die reine und angewandte Mathematik, vol. 49 (1855), pp. 142–160.

[3] Association française pour l'avancement des sciences, vol. 40 (1911), pp. 55–60.

[4] Dirichlet-Dedekind *Vorlesungen über Zahlentheorie*, 4th edition (Braunschweig, 1894), pp. 154–158.

and putting $x = A\alpha + B\gamma$, $y = \gamma$, we have

$$x^2 + ay^2 = kn \qquad \text{with} \qquad k \leqq 2(a/3)^{1/2}.$$

To complete the proof of the theorem it is necessary to show that $ry \equiv \pm x \pmod{n}$. Since T converts (A, B, C) into

$$\left(n, r, \frac{r^2 + a}{n}\right),$$

it follows that

$$A\alpha^2 + 2B\alpha\gamma + c\gamma^2 = n$$

$$A\alpha\beta + B(\alpha\delta + B\gamma) + c\gamma\delta = r$$

$$A\beta^2 + 2B\beta\delta + c\delta^2 = (r^2 + a)/n \text{ with } \alpha\delta - \beta\gamma = 1.$$

Using the first and last of these relations it is easy to verify that

$$(A\alpha + B\gamma) \equiv -(A\alpha\beta + B(\alpha\delta + \beta\gamma) + c\gamma\delta)\gamma, \text{ (modulo } n).$$

Hence $x \equiv \pm ry \pmod{n}$.

Recalling Smith's apparent idea that x and y are prime to each other, we note that, if $A\alpha + B\gamma$ has a factor p in common with γ, then $A\alpha \equiv 0 \pmod{p}$, that is $A\alpha$ has a factor in common with γ, a situation which is not unusual in the relation $A\alpha^2 + 2B\alpha\gamma + C\gamma^2 = n$.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

A PARADOX RESULTING FROM INTEGRATION BY PARTS

By J. L. WALSH, Harvard University.

We start with the usual formula for integration by parts

$$\int u\,dv = uv - \int v\,du,$$

and wish to evaluate the integral $\int dx/x$. We set $u = 1/x$, $dv = dx$, deriving of course $du = -dx/x^2$, $v = x$. The above formula for integration by parts becomes

$$\int \frac{dx}{x} = 1 + \int \frac{dx}{x}.$$

Cancellation of the two integrals then yields the familiar equation $0 = 1$.

QUESTION

56. Proposed by J. E. Trevor, CORNELL UNIVERSITY.[1]

[1] The proposer needs an answer to this question for use in some work on the conditions of stability of certain thermodynamic equilibria.

Let $F(x_1, x_2, x_3, x_4)$ be a homogeneous analytic function of the independent real variables x_1, x_2, x_3, x_4, of degree one, wherefore

$$t \cdot F(x_1, x_2, x_3, x_4) = F(tx_1, tx_2, tx_3, tx_4) ; \tag{1}$$

and let quantities X, y_1, y_2, y_3 be defined by the equations

$$X = x_3 + x_4, \quad y_1 = x_1/X, \quad y_2 = x_2/X, \quad y_3 = x_3/X.$$

When we put $t = 1/X$, the equation (1) becomes $F/X = F(y_1, y_2, y_3, 1-y_3)$, which shall be written

$$F = X \cdot f(y_1, y_2, y_3). \tag{2}$$

Let the four x_i be given independent increments δx_i, whereupon the three y_i obtain increments δy_i, which are equal to zero when the conditions

$$\frac{\delta x_1}{x_1} = \frac{\delta x_2}{x_2} = \frac{\delta x_3}{x_3} = \frac{\delta x_4}{x_4}$$

are satisfied. The corresponding increments of the functions F and f shall be denoted by δF and δf, that is

$$F(x_1 + \delta x_1, \cdots, x_4 + \delta x_4) - F(x_1, x_2, x_3, x_4) = \delta F$$
$$f(y_1 + \delta y, \cdots, y_3 + \delta y_3) - f(y_1, y_2, y_3) \quad = \delta f,$$

and Taylor's expansion shall be written in the notation

$$\delta F = \delta^1 F + \delta^2 F + \cdots, \quad \delta f = \delta^1 f + \delta^2 f + \cdots, \tag{3}$$

where $\delta^n F$, or $\delta^n f$, is the sum of terms of order n in the expansion.

In a note in this MONTHLY, vol. 26(1919), p. 445, it is shown that $\delta^1 F$ is connected with $\delta^1 f$, and that $\delta^2 F$ is connected with $\delta^2 f$, by the relations

$$\frac{\delta^1 F}{X} = \frac{\delta X}{X} f + \left(1 + \frac{\delta X}{X}\right)\delta^1 f, \quad \frac{\delta^2 F}{X} = \left(1 + \frac{\delta X}{X}\right)^2 \delta^2 f. \tag{4}$$

Now since, by (2), F is equal to the function Xf of X and f, and X, f are independent, it follows that

$$\delta F = f\delta X + X\delta f + \delta X \delta f. \tag{5}$$

Elimination of δF, $\delta^1 F$, and $\delta^2 F$ between (4), (5), and the first of equations (3) yields

$$\left(1 + \frac{\delta X}{X}\right)\delta f = \left(1 + \frac{\delta X}{X}\right)\delta^1 f + \left(1 + \frac{\delta X}{X}\right)^2 \delta^2 f + \text{terms in } \delta^3 f, \text{ etc.}$$

It is desired to know whether it is generally true that

$$\frac{\delta^n F}{X} = \left(1 + \frac{\delta X}{X}\right)^n \delta^n f, \quad n > 1.$$

RECENT PUBLICATIONS

Edited by W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Elementarmathematik vom höherem Standpunkte aus. I. By FELIX KLEIN. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen. Band XIV. Dritte Auflage. Berlin, Julius Springer, 1924. viii+321 pages, 125 figures.

Before making its appearance as the fourteenth number of the Courant series, this work of the late Felix Klein appeared in two mimeographed editions, the first in 1908, the second in 1911. The present printed form is the original text with the incorporation of the changes in the second edition and the inclusion of additional footnotes not found in the first or second editions. These latter are enclosed in brackets.

As the title indicates, the book deals with elementary mathematics from the point of view of higher mathematics. It is in the form of a series of lectures, primarily intended for teachers of mathematics in secondary schools and for mature students. Klein and Schimmack had published a report (April, 1907) on the mathematics courses in German secondary schools and their organization. The present work of which the first volume is under review is by way of amplification and expansion of this report.

Klein represented in Germany a movement with respect to the teaching of mathematics such as did Perry in England and the National Committee on Mathematical Requirements, J. W. Young, chairman, in the United States. It was prompted by a desire to vitalize the teaching of the subject by giving the teacher a breadth of view with regard to the interrelationships of the various parts of mathematics that might enable him to inspire in his students a real interest and enthusiasm for mathematical study. Klein's purpose was to give the teacher a panorama or survey of the field from the heights which would both deepen his insight and provide wider vistas. As the most prominent German mathematician of his generation, and equally one of the greatest and most versatile, Klein was well qualified to succeed in just such an undertaking.

The book has a decidedly conversational flavor, quite in the style of a heart-to-heart talk and hence is very readable; it exhibits everywhere the accuracy and authority of one who gained world renown. Homely illustrations abound. The graphical side is stressed, as is attested by the presence of 125 figures in the book itself and emphasis placed on pictorial representation. Much interesting historical matter is included to aid in the presentation of the systematic development of mathematics. From Euclid to Poincaré, from axiomatic foundations to the most practical multiplying machine, the reader

tours through the time-space domain of mathematics. Here is mathematical pedagogy of the highest type, shorn of all pseudo-scientific speculation. Here is mathematical history by one who knows not alone names and dates, but significance and interpretation.

The present volume (I) deals with arithmetic, algebra and analysis. Geometry is reserved for volume[1] II and does not enter within the precincts of this review. Part I begins with the natural numbers and includes the extension of the number concept, some elementary number theory, complex numbers together with quaternions. Part II treats mainly of the solution of algebraic equations, with a maximum attention to graphical methods. Solvability is also considered. Part III in analysis treats the logarithmic, exponential and trigonometric functions, differentiation, integration and Taylor's expansion. Two appendices to this part deal with the transcendence of e and π, and an introduction to point sets.

The text itself was prepared by E. Hellinger which ensures its accuracy. Two short chapters are added by Seyfarth giving the developments in mathematical courses in secondary schools in Germany since 1911 and a list of recent mathematical works of interest to teachers. Throughout the book a bibliography is to be found of the classic mathematical treatises. The typographical errors are very few and unimportant.

As an inspiration for the best kind of mathematical teaching this book is recommended to all teachers of mathematics of high school and elementary college grade.

H. J. ETTLINGER.

Éléments de la Théorie des Probabilités. By ÉMILE BOREL. Third Edition. Paris, J. Hermann, 1924. vii+226 pages.

This book deals with discontinuous probabilities, continuous probabilities, and the probabilities of causes, four or five chapters being devoted to each subject. The above mentioned parts appear in this edition as books I, II, and III. The first edition, which appeared in 1909, contained these three books, and the only essential change in the third edition is the inclusion of 39 pages containing four notes. The first note is on radioactivity, probability and determinism; the second, on the employment of the differential method for the comparison of statistics; the third, on the expression of probabilities in certain games of chance in terms of integral equations; and the last, on games of chance as affected by the ability of the players.

To go a little more into detail, the first book deals with *a priori* methods of probability and the theory is illustrated by the usual exercises in coin tossing.

[1] Volume II has already appeared. ED.

The definition of mathematical expectation is given and total and compound probability are illustrated by drawings from urns. The first and second approximations of the formula of Stirling are developed and the normal probability function is defined and a number of its relations are developed. The law of great numbers receives considerable attention in this book, in fact, the book leads up to and concludes with the development of this subject.

The second book is occupied with geometrical probability which is illustrated by numerous examples well known in the classical theory of this subject. The author also deals with arbitrary functions and makes an application to the distribution of the minor planets and another one to the kinetic theory of gases. The book concludes with a chapter on errors of observation and the method of least squares and the various notions arising in connection with this theory.

The third book introduces Bayes' theorem and it is illustrated by various problems. The first approach to modern statistics as developed through empirical processes appears in this book. The notions of empirical probability are illustrated by means of urn schemata and by statistics relating to masculine and feminine births, births of twins, the number of births in a family, and mortality statistics. The book concludes with a brief discussion of continuous probabilities as applied to the determination of causes. The author considers in this connection the distribution of the stars upon the celestial sphere, the values of atomic weights and some biometric applications.

One leaves this book with the general feeling that the author has explained the simple ideas of *a priori* and empirical probability and illustrated them up to a certain point, but falls short of bringing the student to a realization of how much is behind—or perhaps one should say ahead—of these elementary matters which have been set forth. The more extensive applications to large subjects are not sufficiently developed and the reader is left in a state of doubt as to what it is all about. All this of course merely emphasizes the fact that the subject has so many ramifications and applications that if one attempts to circumscribe them all in a relatively small book the general effect must be that of sketchy outline lacking tremendously in illuminating and guiding details. The point I wish to make is well illustrated by the proposed new[1] publication of four volumes and seventeen sub-titles,—in reality seventeen books, of which four have already appeared—dealing either with theory or application. Each one of these seventeen books is about as bulky as the one under review and is devoted to a topic which perhaps in the latter is covered in three or four pages or omitted altogether.

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[1] *Traité du Calcul des Probabilités et de ses Applications.* Par Émile Borel, avec collaborateurs.

I might conclude by saying that the author affords the student a fair start in the elements of the theory of probability and perhaps did as well as anyone could do. The fact remains, however, that any book on this subject which attempts to cover so much ground by way of suggestion and an occasional application—where often the real significance of the application is missed—is almost bound to fail if its object is to leave the reader in a lucid state of mind concerning the fundamentals of the theory of probability and its very great power in varied fields of application.

J. W. GLOVER.

Curve Sghembe Speciali Algebriche e Trascendenti. Volume two. Curve trascendenti. By GINO LORIA. Bologna, N. Zanichelli, 1925. 255 pages. Price L 50.

There are several extensive works on the properties of particular plane curves, among them this author's excellent Special Plane Curves.[1] No correspondingly complete treatment of particular space curves was available before the appearance of this work. We welcome it accordingly as filling a real need.

Although this second volume[2] is entitled Transcendental Curves, its first chapter deals with algebraic curves of arbitrary order. This is followed by a chapter on spherical curves, after which comes a discussion of curves defined by their intrinsic equations. The last chapter is devoted to curves lying on given surfaces. The book is crammed with varied, useful, and well selected information on a wide variety of topics. The chief theorems are proved in full for each type of curves. Numerous (but not exhaustive) references in footnotes serve to guide the reader who wishes to pursue any subject further. The incidental discussion of special surfaces is considerable and adds to the value of the book. The author's wide reading and thorough mastery of his subject is evident on every page.

Minor errors are rather frequent. The introduction of new topics and the statement of restrictive hypotheses are sometimes inadequately displayed.

An index of names for the entire work appears at the end of this volume. The absence of a subject index seems unpardonable in a work of this kind but its place is filled in part by a good table of contents.

C. H. SISAM.

Great Circle Sailing. By L. M. BERKELEY. New York, White Book and Supply Co., 1925. vi+45+25 pages. Price $1.50.

The writer proposes in this pamphlet to show that the advantage of following a great circle track is greater than is commonly supposed, and to encourage

[1] *Spezielle algebraische und transcendente ebene Kurven*, two volumes, Fritz Schütte, 1910–1911.

[2] A review of the first volume will be found in the Bulletin of the American Mathematcal Society, vol. 31, p. 557.

the use of great circle sailing by offering to navigators a method of computing a course that is simpler than any in common use, and a method of correcting the course from time to time, that is not commonly used. He specifically offers a gain of five or six hours on each trans-Atlantic trip as a reward for following a certain set of formulas.

Undoubtedly, increase in the expense of rapid ocean transit and a rising valuation of passengers' time make it more and more important to choose the quickest course, but it is not clear that the practice of the managers of steamship lines is more than normally belated in following the changing conditions. The great circle tracks that would make the greatest gain for the swift liners are precisely those that are most blocked by land and hindered by difficult northern waters; some cannot be sailed, and in others, the loss through lessened speed, battered ships and distressed passengers would offset the greater length of the more southerly passage. These liners do follow the northern lanes in the favorable season. In our Navy, certainly the great circle track is well understood, and is followed, when desirable, with no dearth of convenient methods of navigating. For tramp steamers, the problem is not so acute; short runs between ports mean more cargoes and less fuel to be carried, and there are no fixed schedules that will insure the time gained on a run from being swallowed up in time lost in port.

The author's later argument against these views is not sound, but the promises of his introduction should tempt a navigator to read on, if he is unused to great circle sailing and wishes to improve his practice, or if he is more experienced and is attracted by the promise of an easier method of setting his course. The navigator who does read the text will find himself confronted at the start by an unfamiliar notation and by tables (three figure Gaussians) unfamiliar in form and use. In his usual practice, he already has the choice of the Azimuth Tables, from which he can read his course in a minute, as closely as he can use it, or the Hydrographic Office Azimuth Chart, or Captain Weir's, from either of which he can read his course directly; or, if he prefers to compute, he can choose his favorite from the many methods of solving the Altitude-Azimuth sight. This sight, if he is up-to-date, he uses from eight to twelve times a day in the Marc St. Hilaire method of determining a line of position, and its problem is identical with that of the great circle course and distance. It is difficult to feel that he will think the promise of greater simplicity has been kept, for the intensive study and practice of the fundamental sight has evolved so many formulas that hardly any taste can be left ungratified. Neither is he likely to value the method of occasionally correcting his course, for he ordinarily resets his course five or six times a day, when he works out his routine sights, and that is often enough. No calculation of future settings, unchecked by later observations, is likely to improve on the seamanship of an experienced navigator.

There remains the distinctly polemical part of the pamphlet. In this, the reader finds citations from five modern authorities, all belittling the advantage of the great circle track. He is told that the numerical values quoted, if put to the practical test of solution by the correct method, will be found to be so inaccurate as to disprove the contention of the authorities. He may then turn to the last chapter for an exposition of correct comparison of loxodromic and great circle distances; he is destined to be disappointed again, but he will find grounds for guessing why the authorities have been unanimous in disagreement with the author.

It seems probable that the five authorities were content to make their comparisons by means of the first approximation to the figure of the earth—the terrestial sphere. Through an error of judgment, the author fails to do this, apparently not realizing that storms, prevailing winds, ocean currents and the unavoidable inaccuracies of the steersman are of greater consequence than the ellipticity of the earth's meridians. Worse than this, he makes an improper comparison, and makes it incorrectly. His initial procedure is correct; he sets up the differential triangle for the second approximation, the terrestial spheroid, with the element of arc of the meridian as one leg, and finds a correct differential expression for the slope of the loxodrome. (His deduction would have been simpler if he had followed the ordinary practice of introducing the eccentric angle.) He then obtains the second approximation to the loxodromic course. His integration is correct, but furnishes an awkward formula for computation; use of the Binomial Theorem would have replaced his elaborate correction term by the expression $e^2(\sin L_2 - \sin L_1)$, which would have been adequate, especially as he has used a two-figure value of the eccentricity. At this point he makes a blunder; with the improved course, he returns to the sphere for his element of distance, discarding the element of arc of the elliptic meridian. It is difficult to say what unit of length he should use in giving the distance that results from this hybrid approximation; he uses the nautical mile; if he had completed his second approximation consistently, his use of the earth's equatorial radius would have given his result in minutes of equatorial arc; converting these to nautical miles would have offset in great measure the difference between his first and second approximations. The radius of the terrestial sphere is of course so chosen as to secure this effect. A more fundamental error is his comparison of first approximations of great circle distances with what at least purport to be second approximations of the corresponding loxodromes. If he is to do anything with second approximations, he should find the distance and courses of the track on the spheroid having at each point the latitude and longitude of a corresponding point located on the great circle of the terrestial sphere. When this is done, the courses and distances in the two examples that he presents are these:

		Initial Course	Final Course	Distance (naut. mls.)	Advantage of G.C. (naut. mls.)
Ex. 1. Sphere	Great Circle	41° 33′.8	48° 35′.6	5364.5	9.0
	Loxodrome	38°	35′.5	5373.5	
Spheroid	Great Circle	41° 42′.5	48° 42′.4	5360.9	0.2
	Loxodrome	38°	45′.3	5361.1	
Hybrid	Great Circle	— —	— —	———	—
	Loxodrome	38°	45′.3	5386	
Ex. 2. Sphere	Great Circle	50° 11′.3	78° 13′.0	2721.1	89.5
	Loxodrome	74°	6′.6	2810.6	
Spheroid	Great Circle	50° 18′.0	78° 14′.7	2728.1	89.9
	Loxodrome	74°	9′.5	2818.0	
Hybrid	Great Circle	— —	— —	2721	98
	Loxodrome	74°	9′.6	2819	

In the first example, the loxodrome crosses the equator along in the middle part of the run, and there is consequently little difference between the great circle and loxodrome distances; the author does not compare these distances. The second great circle, from Nantucket Shoals to Liverpool, is a characteristic example of a track in moderately high latitudes (it reaches 54.3 N.), showing a considerable difference between the distance run on it and on the loxodrome. The comparison would be more valuable if the direct loxodrome from Sandy Hook were compared with this great circle distance plus the run to Nantucket Shoals. But this great circle is characteristic in another way; it has been started far enough out to clear Long Island, as the author says, and it consequently clears New England (the direct track from New York would go overland until it reached a point well up on the Labrador coast), but after skirting the coast of Nova Scotia, it cuts across land north of Cape Canso, later crossing Newfoundland (from about 47°.4 N, 56°.2 W to 48°.6 N, 53°.0 W), and still later, of course, Ireland.

The two examples indicate that the use of the terrestial sphere causes little error in the comparison of the two kinds of distance; the author makes about sixteen times as great an error in the second example by flirting with the spheroid. It is fair to say that in his disagreement with the five authorities, he was probably wrong in his criticism of their numerical results.

Advantageous great circles are actually rare, especially in those waters where saving of distance would be most highly appreciated. The northern continents, the Antarctic Ocean and the easterly bulge of South America spoil a great many attractive circles, leaving mainly some from Australia, some around the Cape of Good Hope, and some across the Atlantic over which the shortening of the distance is not great. On one occasion a navigator carefully

laid out a great circle track on a long Pacific run, but found when he finished the run, that on account of corrections he had been obliged to make for ocean currents, he had followed a loxodrome almost exactly.

The concluding paragraphs of the pamphlet give estimates of the saving in cost made by following the track across Newfoundland, but the zest has gone from it.

PAUL CAPRON.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Transactions of the American Mathematical Society, volume 28, No. 3, July 1926: "Double binary forms with the closure property" by A. B. Coble, 357–383; "On a class of polynomials in the theory of Bessel's functions" by J. H. McDonald, 384–390; "On the zeros of the function $\beta(z)$ associated with the gamma function" by T. H. Gronwall, 391–419; "Osculating curves and surfaces" by P. Franklin, 400–416; "On Laplace's integral equations" by J. D. Tamarkin, 417–425; "On Volterra's integro-functional equation" by J. D. Tamarkin, 426–431; "On the discriminant of ternary forms and a certain class of surfaces" by A. Emch, 432–434; "On the convergence of certain methods of closest approximation" by E. Carlson, 435–447; "Functions of plurisegments" by A. J. Maria, 448–471; "Sets of postulates for the logic of propositions" by B. A. Bernstein, 472–478; "Multiple-sheeted spaces and manifolds of states of motion" by H. Hotelling, 479–490; "A theory of a general net on a surface" by V. G. Grove, 491–501; "Cubic curves and desmic surfaces" by R. M. Mathews, 502–522; "Expansion problems in connection with homogeneous linear q-difference equations" by M. G. Carman, 523–535; "Some properties of limited continua, irreducible between two points" by W. A. Wilson, 536–553; "Applications of the theory of relative cyclic fields to both cases of Fermat's last theorem" by H. S. Vandiver, 554–560.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3236. Proposed by H. S. Uhler, Yale University.

Find the equation of a circle (if it exists) such that the sum of the squares of the (radial) distances from the points (1, 0), (0, 1), (−1, 0), (0,−1), (4, 3) to the circumference shall be a minimum.

3237. Proposed by Nathan Altshiller-Court, University of Oklahoma.

A variable chord of an ellipse subtends a right angle at the center of the curve. Find the envelope of the chord.

3238. Proposed by Malcolm Foster, Williams College.

Determine the necessary and sufficient condition that the loci of the centers of first and second curvature of a given curve have orthogonal tangents at corresponding points.

3239. Proposed by Alex S. Wiener, Cornell University.

Solve the following simultaneous equations for u, v, w, x, y, and z:

$$(u-a_1)(x-b_1)-(v-b_1)(w-a_1)=0,$$
$$(u-a_2)(z-b_2)-(v-b_2)(y-a_2)=0,$$
$$(w-a_3)(z-b_3)-(x-b_3)(y-a_3)=0, \text{ and}$$
$$u^2+v^2=w^2+x^2=y^2+z^2=r^2.$$

3240. Proposed by Lü Ling, Tsing Hua University, Peking.

On a given base, construct a triangle such that its vertex may be on a given straight line, and the difference between its base angles may be equal to a given angle.

3241. Proposed by F. A. Berger, Washington University.

With a center on the circumference of a given circle an arc of a second circle is drawn which bisects the area of the first circle. Find the ratio of the radii of the two circles.

3242. Proposed by R. S. Underwood, Auburn, Alabama.

A man finds that a pile of cocoanuts is exactly divisible by n after giving an extra nut to a monkey. He takes away $1/n$th of the remaining nuts and leaves the rest. A second man repeats the process with the rest giving one nut to the monkey and taking away exactly $1/n$th of the rest. This is continued and the nth man leaves at the end a pile of nuts which is exactly divisible by n. How many nuts were there at the beginning and how many at the end?

3243. Proposed by J. L. Riley, Ouachita College.

Show that the number of points whose coördinates are integers (exclusive of the origin) contained within the circle $x^2+y^2=m$ is four times the number contained within the area bounded by the line $x=0$ and the hyperbolas

$$y(4x+1)=m \text{ and } y(4x+3)=m.$$ (Eisenstein, Crelle, XXVII, p. 248.)

SOLUTIONS

2831 [1920, 227; 1925, 481]. Proposed by B. J. Brown, Kansas City, Mo.

Given that the series $\sum_0^\infty a_n y^n$ is absolutely convergent when $|y|<1$, prove that the series $\sum_0^\infty a_n(2x\cos\theta-x^2)^n$ may be arranged in powers of x provided that $|x|<2/5$.

Prove that if x lies between 1 and 2 the series $\sum_0^\infty (2x-x^2)^n$ is convergent and consists only of positive terms, but that the series obtained by arranging it in powers of x diverges.

Solution by Otto Dunkel, Washington University.

In the first part of the problem where x may be real or complex, if $|x|\leqq\rho$, then

$$|y| = |2x\cos\theta - x^2| \leqq 2\rho+\rho^2. \tag{1}$$

The function $\rho^2+2\rho$ increases from 0, when $\rho=0$, to 1, when $\rho=2^{1/2}-1>2/5$. Hence the series

$$\sum a_n(2x\cos\theta - x^2)^n \tag{2}$$

is absolutely convergent if $|x|<2^{1/2}-1$, and this is surely true if $|x|\leqq 2/5$. It will now be shown that the series may be rearranged as a power series in x. We know that, if x is small enough, say $|x|<\delta$, we may

rearrange (2) as a power series in x. Now (2) is an analytic function of an analytic function, if $|x|<2^{1/2}-1$ and it may therefore be expanded as a power series in x within this circle. This expansion must coincide with the one within the δ circle, and hence this latter expansion can be extended, if $\delta<2^{1/2}-1$, to the larger circle. This completes the proof of the first part.

In the second part of the problem x is real. The given series is convergent if $|2x-x^2|<1$. The function $2x-x^2$ decreases from 1 at $x=1$ to -1 at $2^{1/2}+1$. Hence the given series is convergent if $1<x<2^{1/2}+1$. If we restrict x to the shorter interval $1<x<2$, then $2x-x^2$ is positive. This series cannot be written as a power series in x if $|x|>1$. For, if the power series in x were convergent for $x=\bar{x}$ where $|\bar{x}|=\rho>1$, it would be absolutely convergent, x real or complex, in the circle $|x|<\rho$. In this case $1/[1-(2x-x^2)]=1/(1-x)^2$ could be written either as a power series in x or in $(2x-x^2)$ in this circle. It would then follow that $1/(1-x)^2$ is an analytic function in a circle with a radius larger than 1: but this is impossible since there is a singular point at $x=1$. Hence the power series in x cannot converge if $|x|>1$.

3160 [3156; 1926, 46]. Proposed by R. E. Gaines, University of Richmond.

A *given* ellipse always touches a fixed straight line at a given point. Find the locus of its center. Find the locus of the focus of a parabola under similar conditions.

SOLUTION BY ROSCOE WOODS, State University of Iowa.

The locus sought under the conditions of the above problem is known under the name of glissette, that is, a curve traced by a point that is carried by a curve two of whose points glide on a fixed curve. In this case the fixed curve is a straight line and the two points are coincident. One should consult W. H. Besant's *Roulettes and Glissettes*. (This book is not accessible to the writer.) The method used to obtain these loci is general for this type of problem.

Consider the ellipse $x^2/a^2+y^2/b^2=1$. Take any point P ($a\cos\theta$, $b\sin\theta$) on the curve. The equations of the tangent and normal at this point are

$$bx\cos\theta+ay\sin\theta-ab=0,$$

and similarly

$$ax\sin\theta-by\cos\theta-a^2e^2\sin\theta\cos\theta=0,$$

where $a^2e^2=a^2-b^2$.

We may now easily calculate the distances from the origin to these lines. Call the distances Y, X, respectively. Then we find that $X=\pm(a^2e^2\sin\theta\cos\theta)/(b^2+a^2e^2\sin^2\theta)^{1/2}$, $Y=\pm ab/(b^2+a^2e^2\sin^2\theta)^{1/2}$. Consider now the tangent and the normal as fixed lines. Then (X, Y) are the coordinates of the center of the ellipse referred to this pair of lines as axes. If θ is eliminated from these equations, the equation of the center of the ellipse will be given. This elimination gives $X^2Y^2=(a^2-Y^2)(Y^2-b^2)$.

This quartic consists of two loops symmetrical to the axes and origin. The Y-intercepts are $\pm a$, $\pm b$. For values of Y such that $|a|>Y>|b|$, X is real. The loop obtained depends upon which side of the Y-axis the ellipse is placed.

Consider now the parabola, $y^2=4px$ with the focus $(p, 0)$. In precisely the same way as before we can obtain the locus of the focus of the parabola that is always tangent to the y-axis at the origin. Consider the point $(pt^2, 2pt)$. The equations of the tangent and normal at this point are

$$ty-x-pt^2=0,$$

$$y+tx-2pt-pt^3=0.$$

Calculating now the distances from the focus $(p, 0)$ to these lines, we have $Y=\pm p(1+t^2)^{1/2}$, $X=\pm pt(1+t^2)^{1/2}$. The elimination of t from the equations gives the locus of the focus to be $p^2X^2=Y^2(Y^2-p^2)$.

This quartic cuts the Y-axis at the points $(0, \pm p)$ and is symmetrical to the axes and origin. X is real for all values of Y such $Y>|p|$. This curve has the origin as a conjugate point.

Also solved by E. F. ALLEN, THEODORE BENNETT, E. M. BERRY, RUFUS CRANE, ALEXANDER DILLINGHAM, and WILLIAM ROTH.

3161 [3157; 1926, 47]. Proposed by Dr. Bernardo Ig. Baidaff, Buenos Aires, Argentina.

Calculate

$$\underset{p}{(}x\,\underset{p-1}{(}x\cdots\underset{2}{(}x\,\underset{1}{(}xe^x)'\,\underset{2}{)}'\cdots\underset{p-1}{)}'\,\underset{p}{)}',$$

$$\underset{p}{(}x\,\underset{p-1}{(}x^2\cdots\underset{3}{(}x^{p-2}\underset{2}{(}x^{p-1}\underset{1}{(}x^p)'\,\underset{1}{)}\underset{2}{)}''\,\underset{3}{)}'''\cdots\underset{p-1}{)}^{p-1}\,\underset{p}{)}^{(p)}$$

where the accents $'$, $''$, $\cdots$, mean the first, second, $\cdots$, derivatives, and where the parentheses are paired as indicated by the numbers under them.

SOLUTION BY THEODORE BENNETT, University of Illinois.

Let $\phi_0=e^x$, $\phi_1=(xe^x)'$, $\phi_2=(x(xe^x)')'$, $\cdots$, so that ϕ_i is the result obtained by the i-th differentiation. Then we have

$$\phi_{i+1} = (x\phi_i)'. \tag{1}$$

It is evident that ϕ_i is the product of e^x and a polynomial in x of degree i; moreover, the coefficient of x^i and the constant term in this polynomial are both equal to unity. Therefore, let us write

$$\phi_i = e^x \sum_{j=0}^{i} {}_iu_j x^j, \tag{2}$$

where we know that ${}_iu_0={}_iu_i=1$. Using this expression for ϕ_i (and the corresponding one for ϕ_{i+1}), equation (1) becomes

$$e^x \sum_{j=0}^{i+1} {}_{i+1}u_j x^j = \frac{d}{dx}\left[\, e^x \sum_{j=0}^{i} {}_iu_j x^{j+1}\right] = e^x \sum_{j=0}^{i+1} [(j+1){}_iu_j + {}_iu_{j-1}]x^j.$$

Therefore, ${}_iu_j$ is such a function of i and j that

$${}_{i+1}u_j = (j+1){}_iu_j + {}_iu_{j-1}, \quad j = 0,1,2,3,\cdots,i+1, \tag{3}$$

where we know in advance that ${}_iu_0={}_iu_i=1$, and ${}_iu_j=0$ if $j>i$ or $j<0$.

If we write the first $j+1$ equations of this set (obtained by setting $j=0, 1, 2, \cdots, j$) and from them eliminate all the u's with right hand subscripts less than j, we arrive at a linear difference equation of order j with constant coefficients, i.e. they involve j but not i. We solve this equation by a method analogous to that of solving a linear differential equation with constant coefficients. We shall not give the details of this work (which would be a bit tedious) but shall state the result and verify that it is correct. This result may be written

$${}_iu_j = \frac{1}{j!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (j-r+1)^i. \tag{4}$$

We propose to show by mathematical induction that (2) gives the correct value of ϕ_i provided ${}_iu_j$ has the value given by (4). If $i=1$ we have from (4) ${}_1u_0=1$, ${}_1u_1=1$, which is correct, since by direct differentiation $\phi_1=(xe^x)'=e^x(1+x)$. To complete the proof by induction we have only to show that ${}_iu_j$ as given by (4) satisfies (3). Substituting in (3) we have

$$\frac{1}{j!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (j-r+1)^{i+1} = (j+1)\frac{1}{j!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (j-r+1)^i$$
$$+ \frac{1}{(j-1)!} \sum_{t=0}^{j-1} (-1)^t \binom{j-1}{t} (j-t)^i. \tag{5}$$

The terms of the first two sums for which $r=0$ balance each other; omitting these two terms and rewriting the third sum in the form

$$\frac{1}{(j-1)!} \sum_{r=1}^{j} (-1)^{r-1} \binom{j-1}{r-1} (j-r+1)$$

we see that (5) is an identity, since

$$\frac{1}{j!} \binom{j}{r}(j-r+1) \equiv \frac{1}{j!}(j+1)\binom{j}{r} - \frac{1}{(j-1)!}\binom{j-1}{r-1}.$$

Hence we see that ϕ_i is given correctly by (2) and (4). The expression written in the statement of the problem is ϕ_p.

From the numbers ${}_iu_j$ we may form a triangular array which is similar to Pascal's triangle, and in fact the easiest way to compute ϕ_i for small values of i is to form this array, using equation (3). It is interesting to note that in the Pascal triangle the equation corresponding to (3) is ${}_{i+1}u_j = {}_iu_j + {}_iu_{j-1}$.

To evaluate the second expression given in the problem we let

$$\phi_0 = 1, \quad \phi_1 = (x^p)', \quad \phi_2 = (x^{p-1}(x^p)')', \cdots$$

so that

$$\phi_{i+1} = (x^{p-i}\phi_i)^{(i+1)}. \tag{6}$$

It is fairly evident that ϕ_i is a constant multiplied by the $i(p-i)$th power of x; accordingly we assume

$$\phi_i = C_i x^{i(p-i)}. \tag{7}$$

Substituting this value of ϕ_i in (6) we have

$$C_{i+1}x^{(i+1)(p-i-1)} = (C_i x^{(i+1)(p-i)})^{(i+1)} = C_i \frac{[(i+1)(p-i)]!}{[(i+1)(p-i)-(i+1)]!} x^{(i+1)(p-i-1)}.$$

Hence we wish to find C_i so that

$$C_{i+1} = \frac{[(i+1)(p-i)]!}{[(i+1)(p-i-1)]!} C_i. \tag{8}$$

Then evidently

$$C_i = \prod_{j=1}^{i} \frac{[j(p-j+1)]!}{[j(p-j)]!}. \tag{9}$$

Then from (7) we see that the desired quantity, namely ϕ_p, is given by (9) for $i=p$.

3168 [3164; 1926, 104]. Proposed by R. H. Sciobereti, University of California.

Study qualitatively the variations of the function

$$y = 2xe^{(1/x)} + 4x^3 - 15x^2 + 18x,$$

(the numerical values of the extrema are not required).
Find a polynomial $F(x)$ such that the difference $y-F(x)$ approaches zero when x becomes infinite.

SOLUTION BY THE PROPOSER.

The function y is defined and continuous for any value of x, except zero. Since the polynomial $4x^3-15x^2+18x$ vanishes only for $x=0$, it follows that y is negative for $x<0$ and positive for $x>0$. At $x=0$, the function being discontinuous, let us find limit y when x approaches zero; then

$$\lim_{x\to 0} y = \lim_{x\to 0} 2xe^{1/x} = \lim_{x\to 0} 2e^{1/x};$$

hence the limit of y is either $+\infty$ or 0, according as x approaches 0 through positive or negative values. Let us now determine the extrema of the function by finding the roots of the derivative

$$\tfrac{1}{2}y' = (x-1)(e^{1/x}\cdot x^{-1} + 6x - 9).$$

For $x<0$, the derivative is always positive; for $x=0$, it is discontinuous, while for $x>0$ the discussion is somewhat more complicated and the roots of the expression $u = e^{1/x}\cdot x^{-1}+6x-9$ must first be found. Denoting by ϵ a positive infinitesimal, we shall have $u(\epsilon)=+\infty$, $u(1)=e-3<0$, $u(\tfrac{1}{2})=\tfrac{3}{2}e^{2}>0$; therefore the function $u(x)$ which is continuous for $x>0$ vanishes an odd number of times in each of the intervals (0,1) and $(1,\tfrac{1}{2})$. The first two derivatives of u are $u'=-x^{-2}\cdot e^{1/x}(x^{-1}+1)+6$ and $u''=x^{-4}\cdot e^{1/x}(2x+4+x^{-1})$. We readily see that u' vanishes only once between 0 and 1, since $u'(\epsilon)=-\infty$, $u'(1)=-2e+6>0$ and since u' is an increasing function of x, the second derivative u'' being always positive for $x>0$. Consequently the function $u(x)$ vanishes but once for $x=x_1$ in the interval (0,1) (Rolle's theorem). In the same manner it can also be shown that $u(x)$ vanishes only once for $x=x_2$ in the interval $(1,\tfrac{1}{2})$. It follows that the derivative y' vanishes and changes sign at the points $x=x_1$, $x=1$ and $x=x_2$; hence

the function y has a relative maximum at $x=1$ and two relative minima at $x=x_1$, $x=x_2$. The other special values of the first derivative and the sign of the second derivative $\frac{1}{4}y''=x^{-3}\cdot e^{1/x}+12x-15$ are easily determined and will be given in the following table:

x	$-\infty$		$-\epsilon, 0, +\epsilon$		x_1		1		x_2	3/2		$+\infty$
y''	$-\infty$	$-$	$-30\vert+\infty$	$+$		$+0-$	$e-3$	$-0+$			$+$	$+\infty$
y'	$+\infty$	$+$	$18\vert-\infty$	$-$	0	$+$	0	$-$	0	$+$		$+\infty$
y	$-\infty$	$-$	$-\eta\vert+\infty$	$+$	min.	$+$	max.	$+$	min.	$+$		$+\infty$

The polynomial $F(x)$ such that $\lim_{x\to\infty}[y-F(x)]=0$ is evidently of the form $4x^3-15x^2+2ax+2b$, and the coefficients a and b will now be determined by the condition

$$\lim_{x\to\pm\infty} 2[x(e^{1/x}+9-a)-b]=0, \text{ or } \lim_{x\to\pm\infty}\frac{e^{1/x}+9-a}{1/x}=b,$$

which shows that the quantity $(e^{1/x}+9-a)$ must be an infinitesimal of the same order as $1/x$ (b being of course finite and $\neq 0$); hence $\lim_{x\to\pm\infty}(e^{1/x}+9-a)=0$, which gives $a=10$ and therefore $\lim_{x\to\pm\infty}(e^{1/x}-1)\div 1/x=1=b$, so that $F(x)=4x^3-15x^2+20x+2$. This shows that the curve representing the function $y=2x\,e^{1/x}+4x^3-15x^2+18x$ is asymptotic to the cubic $y=4x^3-15x^2+20x+2$, the curve being above the cubic for $x=+\infty$ and below for $x=-\infty$.

3170 [3166; 1926, 104]. Proposed by A. A. Bennett, Lehigh University.

Given an isosceles triangle ABC, in which $AC=BC$, and a circle with center at C. Find a point, P, on the circle such that the tangent to the circle at P bisects the angle APB.

I. Solution by Velma Maness, Oklahoma University.

Let the circumcircle of ABC cut the given circle in the points P and P_1, and let CD be a diameter of the circumcircle. Then, since CD is perpendicular to AB, arc AD = arc BD, and hence angle APD = angle BPD. Moreover, PD is a tangent to the given circle at P. Hence P and P_1 are the desired points. It is here assumed that the radius of the given circle does not exceed the length of the equal sides, otherwise no solution is possible for internal bisectors. Two solutions for external bisectors are always possible by taking the points of intersection of the given circle with CD.

II, III, Solutions by Nathan Altshiller-Court, University of Oklahoma.

We shall generalize the problem by assuming that the given triangle ABC is scalene, i.e. the perpendicular to the line AB from the center C of the given circle (C) does not necessarily pass through the midpoint of the segment AB. Furthermore, we shall consider that a point P of (C) satisfies the conditions of the problem, if either the internal or the external bisector of the angle APB is tangent to the circle (C).

Let P be a required point on the given circle (C), and PM, PM' the bisectors of the two angles formed by the lines PA, PB. The traces M, M' of these bisectors on the line AB are harmonically separated by the points A, B. If PM is the tangent to (C) at P, then PM' passes through the center C of (C). Moreover, the polar m of M passes through the pole S of the line AB with respect to (C) and meets AB in the conjugate N of M in the involution of conjugate points on AB with respect to (C). The point P is thus the intersection of the polar m of M with the line joining C to M'.

If M is made to vary on AB we have

$$C(M'\cdots)\overline{\wedge}(M'\cdots)\overline{\wedge}(M\cdots)\overline{\wedge}(N\cdots)\overline{\wedge}S(N\cdots);$$

hence, the point $P\equiv(CM', SN)$ describes, in general, a conic (H) passing through the points C and S. Two other points of (H) are given by the intersections of the lines CA and CB with the circle having CS as a diameter. The tangent to (H) at C passes through the midpoint of the segment AB.

The polar $m \equiv SN$ of M with respect to (C) is perpendicular to MC, hence CM' will be parallel to SN, if CM is perpendicular to CM'. Thus (H) will have a real point at infinity, if a segment MM' can be found harmonic to the segment AB and subtending a right angle at C. Such a segment is determined on AB by the bisectors of the angles formed by the lines CA, CB. When M coincides with either end of this segment, we will obtain a point at infinity of (H). Thus the locus of P is an equilateral hyperbola whose asymptotes are parallel to the bisectors of the angles formed by the lines CA, CB.

The hyperbola (H) meets the circle (C), in general, in four points, hence the problem has four solutions. Since (H) passes through the center C of the circle (C), two of these solutions are always real, the other two may be real or conjugate imaginary.

Now if we suppose, as it is done in the proposed problem, that the points A, B are symmetric with respect to the line CS, the hyperbola (H) will degenerate into two straight lines, one of which is the perpendicular bisector CS of the segment AB, and the other a line parallel to AB. It is readily seen that this parallel line is the common chord of the circle (C) and the circumcircle of the triangle ABC.

The reader may consider the case when the points A, B, C are collinear.

If one of the bisectors PM, PM', say, PM of the angles formed by the lines PA, PB is tangent to the given circle (C), then the other bisector PM' passes through the center C of (C).

The bisectors PM, PM' are the tangents to the two conics passing through P and having their foci at the points A, B. Thus a point P satisfying the conditions of the problem is the point of contact of a tangent drawn from the given point C to a conic of the confocal pencil of conics having for their foci the points A, B. Now it is known that the locus of the points of contact of the tangents drawn from a fixed point to the conics of a homofocal system with foci at A, B is a circular cubic (F) passing through A, B and having a double point at C, the tangents to (F) at C being the bisectors of the angle ACB. The four finite points of intersection of (F) with the circle (C) constitute the solution of the proposed problem. Two of these points, namely the points of intersection of the circle (C) with the two infinite branches of (F) are always real. The other two points of intersection lie on the node of (F), and may be real or imaginary conjugate, depending upon the relative size and position of the circle and the segment AB.

Also solved by J. P. ALBERT, E. F. ALLEN, H. C. BRADLEY, THEODORE BENNETT, A. G. CLARK, P. A. CARIS, ALICE A. GRANT, MICHAEL GOLDBERG, F. H. MILLER, W. J. PATTERSON, R. S. SHAW, H. S. UHLER, ROSCOE WOODS, and the PROPOSER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor W. D. CAIRNS, the secretary-treasurer of the Mathematical Association, is spending the year in California, on leave of absence from Oberlin College. He will be at Berkeley until early in May and then at the University of California at Los Angeles until early in August. The Association business is being carried on by the assistant secretary, Professor C. H. Yeaton, at Oberlin.

The Third Carus Monograph, on "Mathematical Statistics," by Professor H. L. RIETZ, is in type but the press work and binding will delaythe publication till sometime in March.

The "Rhind Mathematical Papyrus" is also delayed on account of final work on the hieroglyphic plates by the Egyptologist. All orders are being filed and delivery is anticipated not later than April ist.

Dr. HELEN BARTON has been appointed professor of mathematics and head of the department at Alabama College, which is a State College for women at Montevallo, Alabama.

Mr. N. C. GRIMES has been appointed professor of mathematics and head of the department at Grove City College, Grove City, Pennsylvania, to succeed Professor ARTHUR RAMSAY who resigned at the end of the last college year.

At Wheaton College, Wheaton, Illinois, Mr. C. R. HILLIARD has been appointed professor of mathematics. He was formerly registrar and professor of mathematics at the College of Ozarks. Professor H. O. TAYLOR (Ph.D., Cornell) is teaching part time during the second semester of this college year.

Mr. L. G. BUTLER has been appointed professor of mathematics at Albany College, Albany, Oregon.

Dr. D. R. DAVIS has been appointed assistant professor of mathematics at the University of Oregon.

Mr. P. S. DWYER has been appointed assistant professor of mathematics at Antioch College.

Dr. W. W. ELLIOTT has been appointed assistant professor of mathematics at Duke University.

Associate Professor M. G. GABA, of the University of Nebraska, has been promoted to a full professorship of mathematics.

Dr. MAYME I. LOGSDON, of the University of Chicago, has been promoted to an assistant professorship of mathematics.

Assistant Professor F. S. NOWLAN, of the University of Manitoba, has been appointed professor of mathematics at the University of British Columbia

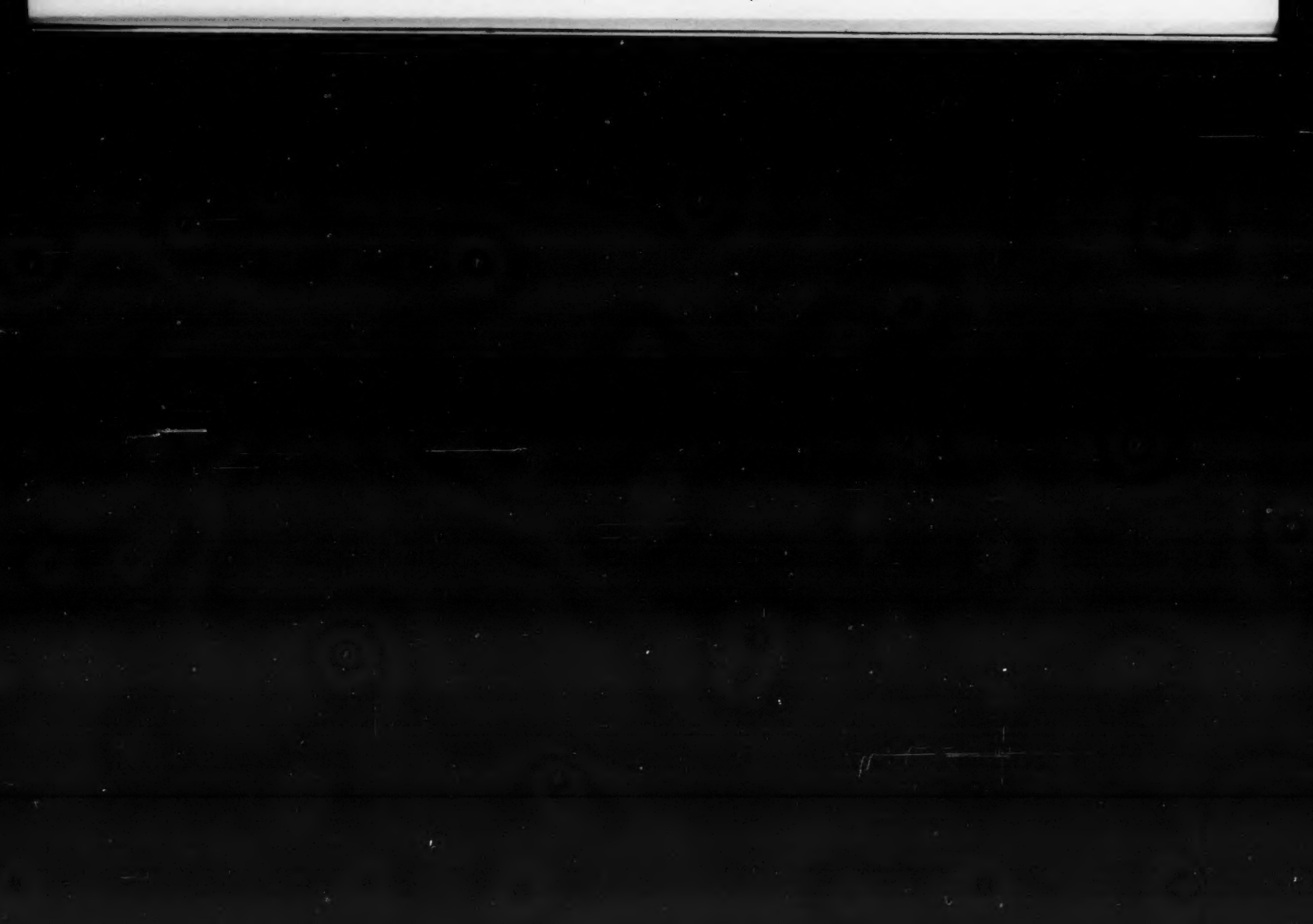
The following appointments to instructorships in mathematics are announced: Columbia University, Dr. Edgar DEHN; Lehigh University, Mr. George RIDDLE; Yale University, Mr. T. W. MOORE.

Professor E. BLASCHKE, of the Vienna Technical School, died October 30, 1926, at the age of seventy.

Professor ALEXANDER TSCHUPROW, formerly of the Technical Institute at Petrograd, died April 19, 1926, at the age of fifty-two. He was known for his work in mathematical statistics.

Professor LOUIS SIFF, of Louisville University, died December 25, 1926, at the age of fifty-seven.

Professor J. H. TUDOR, of Pennsylvania State College, died June 7, 1926, at the age of sixty-nine. Professor TUDOR was a member of this Society.



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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 6-7, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 6-7.

INDIANA, De Pauw University, May 6-7.

IOWA, University of Iowa, April 29-30.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, May.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, Neb., May 14, 1927.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November.

ROCKY MOUNTAINS, Colorado College, April 22-23.

SOUTHEASTERN, March.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATION: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

The Carus Mathematical Monographs

The CARUS MONOGRAPHS have already begun to fulfill their mission as intended by the generous donor, MRS. MARY HEGELER CARUS and her son, DR. EDWARD H. CARUS.

The first edition of Monograph Number One, "Calculus of Variations," by Professor Bliss, is exhausted and a new edition is being printed.

Monograph Number Two, "Analytic Functions of a Complex Variable," by Professor Curtiss, is running practically the same as Number One during its first year.

Over 1000 members of the ASSOCIATION have taken advantage of the distribution of these Monographs at cost. Those who neglected to do so at the start may still have the privilege by applying to the *Secretary*.

Monograph Number Three, "Mathematical Statistics," by Professor Rietz, is in the hands of the printer and will be ready early in 1927.

It would be a great tribute to the donor and an honor to the ASSOCIATION if *every member* would subscribe for the complete series. Every dollar thus paid by members is a contribution to the permanent CARUS PUBLICATION FUND of the ASSOCIATION.

It is believed that the ASSOCIATION is rendering a great service to mathematics by this enterprise, and a one hundred per cent support from the membership would constitute an appropriate vote of confidence in the undertaking.

Orders from members, individual or institutional, should be sent to the Secretary at Oberlin, Ohio. All distribution to the general public will be made through

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339 E. Chicago Ave., Chicago, Illinois

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE ELEVENTH ANNUAL MEETING OF THE ASSOCIATION

The eleventh annual meeting of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Thursday and Friday, December 30–31, 1926, in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. Two hundred and sixty-four were present at the various sessions, including the following one hundred and eighty-seven members of the Association:

C. R. Adams, Brown University.
V. W. Adkisson, University of Pennsylvania.
R. C. Archibald, Brown University.
C. S. Atchison, Washington and Jefferson College.
W. L. Ayres, University of Pennsylvania.

Clara L. Bacon, Goucher College.
Ida Barney, New Haven, Conn.
A. A. Bennett, Lehigh University.
William James Berry, U.S. Bureau of Standards.
O. F. H. Bert, Washington and Jefferson College.
E. G. Bill, Dartmouth College.
G. D. Birkhoff, Harvard University.
G. A. Bliss, University of Chicago.
Henry Blumberg, Ohio State University.
J. W. Bradshaw, University of Michigan.
B. H. Brown, Dartmouth College.
H. S. Brown, Hamilton College.
Margaret Buchanan, West Virginia University.
R. W. Burgess, Western Electric Co., New York.

Minnie W. Caldwell, Chowan College.
W. B. Campbell, Cornell University.
P. A. Caris, University of Pennsylvania.
Mildred E. Carlen, Brown University.
I. S. Carroll, Syracuse University.
J. A. Clarke, West Philadelphia High School for Boys.
J. W. Clawson, Ursinus College.
A. B. Coble, University of Illinois.
Abraham Cohen, Johns Hopkins University.
J. B. Coleman, University of South Carolina.
J. W. Colliton, High School, Trenton, N. J.
Julia T. Colpitts, Iowa State College.
Rufus Crane, Ohio Wesleyan University.
E. S. Crawley, University of Pennsylvania.